Cotangent Complex II

§1) Properties + Construction of $L_{X/Y}$

Affine case: $A \to B$ ring map.

Let
$$\Sigma_\bullet : \quad \cdots \Sigma_2 \begin{smallmatrix}\rightarrow\\\rightarrow\\\rightarrow\end{smallmatrix} \Sigma_1 \rightrightarrows \Sigma_0 \xrightarrow{\varepsilon} B$$

be a free simplicial A-algebra resln of B,

i.e. i) $\Sigma_i = A[\underline{X_i}]$ for some sets $\underline{X_i}$

ii) $s_r(Y) \in X_{i+1}$ for $Y \in X_i$, s_r a degeneracy

iii) $\cdots \xrightarrow[d]{} \Sigma_2 \xrightarrow[d]{} \Sigma_1 \xrightarrow[d]{} \Sigma_0 \xrightarrow[\varepsilon]{} B \to 0$

is exact, where $d = \sum (-1)^i d_i$

Then $L_{B/A} \in D^-(B)$ is

$$\cdots \to \Omega^1_{\Sigma_2/A} \otimes_{\Sigma_2} B \to \Omega^1_{\Sigma_1/A} \otimes_{\Sigma_1} B \to \Omega^1_{\Sigma_0/A} \otimes B \to 0$$

Observe: Augmentation $L_{B/A} \to \Omega^1_{B/A}[0]$.

Ex (Free A-alg. resln)

$$\cdots \to A[A[B]] \rightrightarrows A[B] \to B$$

(Forget "outer brackets") $[\sum c_i [b_i]] \mapsto \sum c_i [b_i]$ $\quad [b] \mapsto b$

(Forget "inner brackets") $[\sum c_i [b_i]] \mapsto [\sum c_i b_i]$

Exercise 1) Work out the rest of the maps

2) Show this is a resln

(Hint: Weibel section on bar construction)

Non-affine case: $f: X \to Y$

Imitate above construction for map of sheaves of rings $f^{-1}\mathcal{O}_Y \to \mathcal{O}_X$.

Main Thm:

Let $f: X \to Y_0$ be a morphism of schemes and $Y_0 \hookrightarrow Y_1$ a closed embedding defined by an ideal sheaf $\mathcal{I}$ w/ $\mathcal{I}^2 = 0$. Then $\exists\, o(f) \in \mathrm{Ext}^2(L_f, f^*\mathcal{I})$ (functorial etc..) s.t. $\exists$

$$\begin{tikzcd} X \arrow[r, dotted] \arrow[d, "f"'] \arrow[dr, phantom, "\square"] & X' \arrow[d, dotted, "\text{flat}"] \\ Y_0 \arrow[r, hook] & Y_1 \end{tikzcd}$$

iff $o(f) = 0$.

flat, $\underline{0 \to f^*\mathcal{I} \to \mathcal{O}_{X'} \to \mathcal{O}_X \to 0}$

In this case, such deformations are a torsor for $\mathrm{Ext}^1(L_f, f^*\mathcal{I})$, and each deformation has aut. gp $\mathrm{Hom}(L_f, f^*\mathcal{I})$.

Properties of $L_{X/Y}$

(1) If $f: X \to Y$ is smooth, $L_{X/Y} \to \Omega^1_{X/Y}[0]$ is an isomorphism

(2) If $g: Z \hookrightarrow X$ is lci embedding, $L_{Z/X} \to \mathcal{I}_Z/\mathcal{I}_Z^2[1]$ is an isom.

(3) If $X \xrightarrow{f} Y \xrightarrow{g} Z$, $\exists$ distinguished triangle (transitivity triangle)

$$f^*L_{Y/Z} \to L_{X/Z} \to L_{X/Y} \to$$

Ex $f^*\Omega^1_{Y/Z} \to \Omega^1_{X/Z} \to \Omega^1_{X/Y} \to 0$, left exact if f smooth

Ex $X \overset{\mathrm{cl}}{\underset{\iota}{\hookrightarrow}} Y \to Z$

$$N^\vee_{X/Y} \to \iota^*\Omega^1_{Y/Z} \to \Omega^1_{X/Z} \to 0$$ left exact if $X \to Z$ smooth

(4) (Base Change)

$$\begin{tikzcd} X \times_Z Y \arrow[r, "\tilde{g}"] \arrow[d, "\tilde{f}"] \arrow[dr, phantom, "\square"] & X \arrow[d, "f"] \\ Y \arrow[r, "g"'] & Z \end{tikzcd}$$

$\tilde{g}^* L_{X/Z} \to L_{X \times_Z Y / Y}$ natural

isom. if for g flat.

$L_{X \times_Z Y / Z} = \tilde{g}^* L_{X/Z} \oplus \tilde{f}^* L_{Y/Z}$ as well.

§2) Example Application 1: Lifting lci curves

Suppose $X \xrightarrow{f} Y$ factors as
$$\begin{tikzcd} & P \arrow[d, "sm"] \\ X \arrow[ur, "\text{reg. emb.}"] \arrow[r, "f"'] & Y \end{tikzcd}.$$

Then $f^* L_{P/Y} \longrightarrow L_{X/Y} \to L_{X/P} \to$

$f^* L_{P/Y} = f^* \Omega^1_{P/Y}$, $\quad L_{X/P} \simeq N^\vee_{X/P}$

Give example of non-lci thing: $k[x,y,z]/(xy, yz, xz)$ i.e. 3 axes $\subset \mathbb{A}^3$.

hence $L_{X/Y}: 0 \to N^\vee_{X/P} \to f^* \Omega^1_{P/Y} \to 0$

Hence L_f is concentrated in deg $[-1, 0]$ for f lci.

Thm Let X be a gen. sm, lci proj. curve over k. Let R be a complete local Noetherian (m-adic) ring w/ $R/m = k$. Then X admits a lift to R, eg. $\mathcal{X} \to \operatorname{Spec} R$ flat projective w/ $\mathcal{X}_k = X$.

Pf idea Deform, then algebraize.

$$\begin{tikzcd} X \arrow[r, dotted] \arrow[d, "f"'] & X' \arrow[r, dotted] \arrow[d, dotted] & X'' \arrow[r, dotted] \arrow[d, dotted] & \cdots \\ \operatorname{Spec} k \arrow[r, hook] & \operatorname{Spec} R/m^2 \arrow[r, hook] & \operatorname{Spec} R/m^3 \arrow[r, hook] & \cdots \end{tikzcd}$$

Existence of deformations controlled by

$$o(X) \in \operatorname{Ext}^2(L_{X/k}, f^*(m/m^2))$$

Local-global spectral sequence

(First-order deformations — higher order identical)

$$H^i(X, \mathcal{E}xt^j(L_{X/k}, f^*(m/m^2))) \Rightarrow \operatorname{Ext}^{i+j}(L_{X/k}, f^*(m/m^2))$$

$$H^0(X, \mathcal{E}xt^2(L_{X/k}, f^*(m/m^2))) = 0$$

b/c $L_{X/k}$ concentrated in $[-1,0]$, so

$$\mathcal{E}xt^2(L_{X/k}, f^*(m/m^2)) = 0$$

$$H^2(X, \mathcal{H}om(L_{X/k}, f^*(m/m^2))) = H^2(X, \mathcal{H}om(\Omega^1_{X/k}, f^*(m/m^2)))$$
$$= 0$$

b/c X is a curve.

$H^1(X, \mathcal{E}xt^1(L_{X/k}, f^*(m/m^2))) = 0$ b/c

dim supp $\mathcal{E}xt^1(L_{X/k}, f^*(m/m^2)) = 0$

<u>Pf</u> $\mathcal{E}xt^1(L_{X/k}, f^*(m/m^2))\big|_{X^{sm}} = \mathcal{E}xt^1(L_{X^{sm}/k}, f^{sm*}(m/m^2))$

$= 0$ b/c $L_{X^{sm}/k} = \Omega^1_{X^{sm}/k}$ loc. free.

Hence obtain $\begin{array}{c}\mathfrak{X} \\ \downarrow \\ \mathrm{Spf}\, R\end{array}$. (Explain this)

To algebraize, need to lift an ample line bundle.
But obstructions to lifting a l.b. lie in $H^2(\mathcal{O}_X) = 0$.

Maybe write down relevant part of formal GAGA? (EGA 5.4.5) $\square$

This is a generalization of: sm. proj. things w/ $H^2(X, T_X) = 0$ formally lift.

§3) Example Application: Witt Vectors

Want to give conceptual construction of Witt vectors for perfect $\mathbb{F}_p$-algebras.
(i.e. $A/\mathbb{F}_p$ s.t. $\mathrm{Frob}_p : A \to A$ is an isom.)

<u>Lemma</u>. X/S a scheme s.t. $F_{X/S} : X \to X^{(p)}$ is an isom, w/ $\mathrm{Frob}_p : S \to S$ an isom. Then $L_{X/S} = 0$.

Recall:
$$\begin{tikzcd} X \arrow[dr, dotted, "F_{X/S}"] \arrow[drr, bend left, "\mathrm{Frob}_p"] \arrow[ddr, bend right] & & \\ & X^{(p)} \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & X \arrow[d] \\ & S \arrow[r, "\mathrm{Frob}_p"] & S \end{tikzcd}$$

<u>Idea</u>: Frobenius "on the fibers" of $X \to S$.

Pf. Transitivity triangle:

$$F_{X/S}^* L_{X^{(p)}/S} \to L_{X/S} \to L_{X/X^{(p)}} \to$$

$L_{X/X^{(p)}} = 0$ b/c $F_{X/S} : X \to X^{(p)}$ is an isom.,

hence $F_{X/S}^* L_{X^{(p)}/S} \to L_{X/S}$ is isom.

Would suffice to show this map is zero.

By base change, reduce to affine situation:

$$\begin{tikzcd} B \otimes_A A^{(p)} =: B^{(p)} \arrow[rr, "F_{B/A}"] & & B \\ & A/\mathbb{F}_p \arrow[ul] \arrow[ur] & \end{tikzcd}$$

, want to show $B \otimes_{B^{(p)}} L_{B^{(p)}/A} \to L_{B/A}$ is zero.

w/ Frob: $A^{(p)} \to A$ an isom.

Choose $B_\bullet$ a free A-algebra resln of B.

Then $B_\bullet^{(p)} := B_\bullet \otimes_A A^{(p)}$ is a free A-algebra resln of $B^{(p)}$, and

the component-wise Frobenius

$F_{B_\bullet/A} : B_\bullet^{(p)} \to B_\bullet$ induces $F_{B/A} : B^{(p)} \to B$ on π_0.

Thus $B \otimes_{B^{(p)}} L_{B^{(p)}/A} \to L_{B/A}$ induced by:

$\Omega^1_{B_\bullet^{(p)}/A} \otimes_{B_\bullet^{(p)}} B \longrightarrow \Omega^1_{B_i/A} \otimes_{B_i} B$. But $\Omega^1_{B_i^{(p)}/A} \otimes B_i \to \Omega^1_{B_i/A}$

is already zero b/c $d(x^p) = 0$. $\square$

<u>Cor</u> $A/\mathbb{F}_p$ perfect. Then $\exists!$ (up to ! isom) diagram

$$\begin{tikzcd}
A & A_{(2)} \arrow[l, two heads] & A_{(3)} \arrow[l, two heads] & \cdots \arrow[l] \\
\mathbb{F}_p \arrow[u] & \mathbb{Z}/p^2\mathbb{Z} \arrow[l, two heads] \arrow[u] & \mathbb{Z}/p^3\mathbb{Z} \arrow[l, two heads] \arrow[u] & \cdots \arrow[l]
\end{tikzcd}$$

w/ all squares co-Cartesian, all vertical arrows flat.

<u>Pf</u> Deformations controlled by

$$\mathrm{Ext}^i_A(L_{A/\mathbb{F}_p}, -) \qquad i=0,1,2$$

But $L_{A/\mathbb{F}_p}=0$ by lemma, hence deformations

(1) Exist, b/c $\mathrm{Ext}^2(L_{A/\mathbb{F}_p}, -)=0$

(2) Are unique, b/c $\mathrm{Ext}^1(L_{A/\mathbb{F}_p}, -)=0$

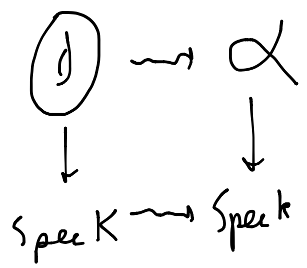
(3) Up to unique isom, b/c $\mathrm{Ext}^0(L_{A/\mathbb{F}_p}, -)=0$. $\square$

<u>Rmk</u> $W(A) = \varprojlim A_{(i)}$

§4) p-adic uniformization of AVs of good reduction.

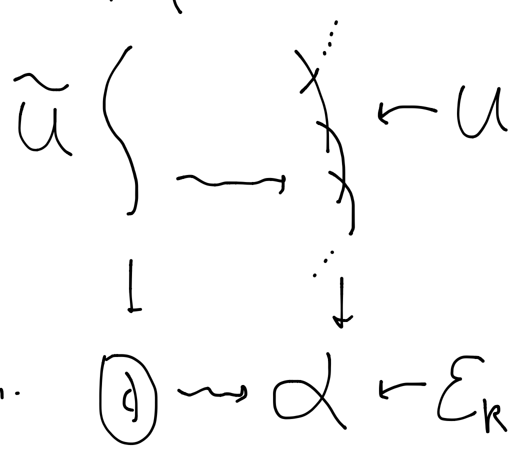
R - complete dvr, w/ res field k, fraction field K

E/K ell. curve of split multiplicative reduction.



Tate: rigid-analytic uniformization of such E:

$$\mathbb{G}_m/\langle q\rangle \twoheadrightarrow E \qquad |q|\neq 1.$$



Cartoon:

$L_{\mathcal{U}/\mathcal{E}_k} = 0$!

Claim: Rigid generic fiber is $\mathbb{G}_m$.
(draw annuli)

Question: Is $\operatorname{Ext}^1(L_{\mathcal{U}/k}, -) = 0$?

If E has good reduction, no obvious "universal cover" of E_k to deform.
Candidate if k perfect of char $p>0$:

$$\varprojlim \left(\cdots \xrightarrow{[p]} E \xrightarrow{[p]} E \xrightarrow{[p]} \cdots\right)$$

<u>Lemma</u> A/k an AV, char $k = p > 0$. Then

$[p]: A \to A$ factors through $F_{A/k}: A \to A^{(p)}$.

<u>Pf</u> (1) $[p] = VF$.

(2) <u>Lemma</u> X integral, normal, X, Y f.t. $/k$.

Suppose $f: X \to Y$ s.t. $f^*\Omega^1_Y \to \Omega^1_X$ is zero.
Then f factors through $F_{X/k}: X \to X^{(p)}$.

<u>Pf</u> Exercise

<u>Lemma</u> $[p]^*\Omega^1_A \to \Omega^1_A$ is zero.

<u>Pf</u> Compute on Lie A. $\square$

<u>Cor</u> Let $\hat{E} = \varprojlim(\cdots \to E \xrightarrow{[p]} E \to \cdots)$. Then
$L_{\hat{E}/k} = 0$.

<u>Pf</u> Suffices to show $F_{\hat{E}/k}$ is an isomorphism.
But $[p]: \hat{E} \to \hat{E}$ is isom, factors through $F_{\hat{E}/k}$.

$$F(V[p]^{-1}) = id, \quad V[p]^{-1}F = id$$

(trivial) (exercise) (use reducedness) $\square$

<u>Cor</u> Let E, E' be ell curves$/\mathbb{Z}_p$
and let $\alpha: E_{\mathbb{F}_p} \xrightarrow{\sim} E'_{\mathbb{F}_p}$ be an isomorphism.
Let $\hat{\tilde{E}}$ be the formal scheme obtained by
completing $\varprojlim(\cdots \to E \xrightarrow{[p]} E \to \cdots)$ at (p),
and similar w/ $\hat{\tilde{E}}'$. Then $\exists!$

$$\begin{tikzcd} \hat{\hat{E}} \arrow[rr, "\tilde{\alpha}"] \arrow[dr] & & \hat{\hat{E}}' \arrow[dl] \\ & \mathrm{Spf}\, \mathbb{Z}_p & \end{tikzcd}$$ extending α.

(Explain what this means.)

Pf Deformations are unique up to unique isom $\square$
b/c $L=0$

Slightly more natural to consider
$u(E) := \varprojlim_{[n]} \hat{E}_p$, can run similar argument
w/ this.

Prop $U(E)$ depends only on isogeny class of E_k.

Prop $\mathrm{Aut}(E_k)_{\mathbb{Q}} \subset \mathrm{Aut}(U(E))$

Open Question: Describe $U(E)_{\mathbb{Z}_p^{ur}}$ in terms of Weil polynomial of E_k.

Rmk Analogous thing for varieties w/ non-trivial Albanese, e.g. curves.

§5) Example Application: Unobstructedness of CYs in char. 0.

<u>Lemma</u> char $k=0$, R a complete local Noeth. k-algebra w/ residue field k. Suppose that for all Artin A/k, M, M' f.g. A-modules w/ $M' \twoheadrightarrow M \to 0$,

$$\mathrm{Hom}(R, A\oplus M') \longrightarrow \mathrm{Hom}(R, A\oplus M)$$

is surjective. Then $R \simeq k[[x_1, \dots, x_n]]$

<u>Pf</u> Can write $R = k[[x_1, \dots, x_n]]/I$, where $n = \dim \mathfrak{m}_R/\mathfrak{m}_R^2$; let $S = k[[x_1, \dots, x_n]]$. May assume $I \subset \mathfrak{m}_S^2$. Suppose $I \subset \mathfrak{m}_S^{n-1}$; wish to show $I \subset \mathfrak{m}_S^n$. Let $R_n = S/\mathfrak{m}_S^n + I = R/\mathfrak{m}_R^n$

Let $A_n = S/\mathfrak{m}_S^{n-1}$, $M_n = \bigoplus_{i=1}^n \varepsilon_i S/\mathfrak{m}_S^{n-2}$, $M_n' = \bigoplus_{i=1}^n \varepsilon_i S/\mathfrak{m}_S^{n-1}$

Then $S \to A_n \oplus M_n$

$$x_i \longmapsto (x_i, \varepsilon_i \cdot 1)$$

factors through $S/\mathfrak{m}_S^{n-1} = R/\mathfrak{m}_R^{n-1}$ b/c e.g.

$$x_{n-1} \mapsto (x_i, \varepsilon_i)^{n-1} = (x_i^{n-1}, (n-1)\, x_i^{n-2} \varepsilon_i)$$

$$f(\underline{x}) \mapsto \left(f(\underline{x}), \frac{\partial f}{\partial x_1}\varepsilon_1 + \cdots + \frac{\partial f}{\partial x_n}\varepsilon_n\right)$$

But this map does not lift to $A \oplus M_n'$ unless $\frac{\partial f}{\partial x_i} \in m_S^{n-1}$ for $f \in I$, i.e. $I \subset m_S^n$. $\square$

this uses char $k=0$

<u>Cor</u> Let $F: \mathrm{Art}_k \to \mathrm{Set}$ be a pro-representable deformation functor, w/ char $k=0$. Suppose that for all $A \in \mathrm{Art}_k$, $M, M' \in$ f.g. A-mod w/ $M' \twoheadrightarrow M \to 0$,

$$F(A \oplus M') \to F(A \oplus M) \text{ is surjective.}$$

Then F is smooth.

<u>Pf</u> Apply Lemma to pro-representing object.

<u>Cor</u> (Tian-Todorov) Calabi-Yau varieties are unobstructed in characteristic zero.

Define CY var.

<u>Pf</u> Let $A \in \mathrm{Art}_k$, $M' \to M \to 0$ a surjection of A-modules, $X/_k$ a CY, and $f: X_A \to \mathrm{Spec}\, A$ a deformation over A. Suffices to show

$$\begin{array}{ccc} \mathrm{Ext}^1(\Omega^1_{X_A/A}, f^*M') & \to & \mathrm{Ext}^1(\Omega^1_{X_A/A}, f^*M) \\ \| & & \| \\ H^1(X_A, T_{X_A/A} \otimes f^*M') & \to & H^1(X_A, T_{X_A/A} \otimes f^* M) \end{array}$$

surjective.

Same as

$$\begin{array}{ccc} H^1(X_A, \Omega^{n-1}_{X_A/A} \otimes f^* M') & \to & H^1(X_A, \Omega^{n-1}_{X_A/A} \otimes f^* M) \\ \| & & \| \\ H^1(X_A, \Omega^{n-1}_{X_A}) \otimes M' & \longrightarrow & H^1(X_A, \Omega^{n-1}_{X_A}) \otimes M \end{array}$$

But by Deligne-Illusie IV, $H^1(X_A, \Omega^{n-1}_{X_A})$ is free over A $\square$.