Arithmetic Restrictions on Geometric Monodromy

$X/\mathbb{C}$ an algebraic variety, $\Lambda = \mathbb{Z}, \mathbb{Z}_\ell, \ldots$

Q: Which reps

$$\pi_1(X(\mathbb{C})^{an}) \to GL_n(\Lambda)$$

arise from geometry?

(i.e. as monodromy rep on a locally constant torsion-free subquotient of $R^i\pi_*\Lambda$, for $\pi: Y \to X$)

number-theoretic interest

Ex $\begin{array}{c} A \\ \downarrow \\ X \end{array}$ an Abelian scheme.

$$\pi_1(X,x) \to GL(T_\ell(A_x))$$

Sample Results:

Thm 1 (L-) Let $A/\mathbb{C}(t)$ be a non-constant AV s.t. $A[\ell]$ is triv. for some odd prime ℓ. Then A has at least 4 pts of bad reduction.

Thm 2 (L-) k-field w/ $k=\bar{k}$. $x_1, \ldots, x_n \in \mathbb{P}^1(k)$. Let $X = \mathbb{P}^1_k \setminus \{x_1, \ldots, x_n\}$. Then $\exists N = N(x_1, \ldots, x_n)$

s.t. if $\rho : \pi_1^{et}(X) \to GL_n(\mathbb{Z}_\ell)$

(1) arises from geometry, and

(2) ρ is trivial mod ℓ^r for some $\ell^r > N$

Then ρ is unipotent. If ρ is pure, it is trivial.

Rem (1) Let $k_0 \subset k$ be the prime field, and let

$$L = k_0\left(\frac{x_a - x_b}{x_c - x_d}\right).$$

Then N only depends on $\mathrm{im}\left(\chi : Gal(\bar{L}/L) \to \hat{\mathbb{Z}}^\times\right)$

cyclotomic char.

(2) Similar results in genus 1, many higher dim'l vars.

(3) Some dependence on L is necessary — if $L = \mathbb{Q}$, no AVs w/ trivial 5-torsion. But for $X(5)$, univ family gives example (here $L = \mathbb{Q}(\zeta_5)$).

(4) This is a real restriction — π_1 is free, so $\exists$ many non-unip reps trivial mod ℓ, which necessarily don't come from geometry.

Thm 2 $\Rightarrow$ Thm 1 b/c cross-ratio must not lie in $\mathbb{Q}$. In particular, true in arbitrary wt.

Motivation:

<u>Torsion conjecture</u>, <u>Geometric Torsion Conjecture</u>

K - # field, (resp. ftn field) of deg (resp. gonality) d. There $\exists N = N(g, d)$ s.t. if A/K is a g-dim'l (traceless) AV, then

$$|A(K)_{tors}| \leq N.$$

Mazur: $K = \mathbb{Q}$, $g = 1 \Rightarrow N = 16$

Merel...: K - # field, $g = 1$.

Poonen...: ftn field, $g = 1$ (gonality of modular curves $\to \infty$)

$g > 1$: Some results (Nadel, Hwang-To, Bakker-Tsimerman...) for ftn fields in char 0. (Analytic methods — hyperbolicity of $\mathcal{A}_g$) Compare + contrast

Goal: Study geometric questions via "anabelian" methods, + in higher wt.

I.e. suppose $X/_k$ a variety,

$$\begin{tikzcd} \pi_1^{ét}(X_{\bar{k}}) \arrow[r, "\rho"] \arrow[d] & GL_n(\mathbb{Z}_\ell) \\ \pi_1^{ét}(X_R) \arrow[ur, dotted] & \end{tikzcd}$$

for some $R/\mathbb{Z}$ finitely generated. What restrictions does this place on ρ?

Antecedent: Groth. pf of quasi-unipotent local mono-dromy thm. This is global version.

Most naive approach:

* • specialize to finite field of char $\neq \ell$, or

• pass to ℓ-adic field (in the paper, WIP)

§2) $G_{\mathbb{F}_q}$ - action on the fundamental gp.

$X/_{\mathbb{F}_q = k}$ sm. w/ snc compactification, ℓ prime to q.

$\bar{x}$ - geom. pt of X, $\pi_1^{ét,\ell}(X_{\bar{k}}, \bar{x})$ - pro-ℓ fund. gp.

$$1 \to \pi_1^{ét}(X_{\bar{k}}, \bar{x}) \to \pi_1^{ét}(X, \bar{x}) \to G_k \to 1$$

$$\rightsquigarrow \quad G_k \to \mathrm{Out}(\pi_1) \to \mathrm{Out}(\pi_1^{\ell})$$

<u>Defn</u> $\mathbb{Z}_\ell[[\pi_1^{ét,\ell}(X_{\bar{k}}, \bar{x})]] := \varprojlim_{\substack{\pi_1 \twoheadrightarrow H \\ \text{cont.}}} \mathbb{Z}_\ell[H]$

R-$\mathbb{Z}_\ell$-algebra, define

$$R[[\pi_1^{ét,\ell}(X_{\bar{k}}, \bar{x})]] := \varprojlim \left(\mathbb{Z}_\ell[[\pi_1^{ét,\ell}(X_{\bar{k}}, \bar{x})]] / \mathcal{I}^n \otimes R\right)$$

where $\mathcal{I}$ is augmentation ideal.

(pro-unipotent completion if $R=\mathbb{Q}_\ell$)

$G_k \to \mathrm{Out}(\pi_1^{et,\ell})$ induces G_k-action on $\mathcal{I}^n/\mathcal{I}^{n+1}$

<u>Prop</u> $(R=\mathbb{Q}_\ell)$ (1) $\mathcal{I}/\mathcal{I}^2 \simeq \pi_1^{\ell,ab}\otimes\mathbb{Q}_\ell \simeq H^1(X_{\bar{k}},\mathbb{Q}_\ell)^\vee$

(2) Wts in $\mathcal{I}^n/\mathcal{I}^{n+1} \in [-n,-2n]$

<u>Rmk</u> Hence G_k acts semi-simply on $\mathcal{I}^n/\mathcal{I}^{n+1}$

Now suppose $\bar{x}$ comes from a rat'l pt, i.e. choose $k \hookrightarrow \bar{k}$ and

$$\begin{array}{c} X \\ \downarrow \uparrow x \\ \mathrm{Spec}\, k \end{array}$$

Then,

$$1 \to \pi_1(X_{\bar{k}},\bar{x}) \to \pi_1(X,\bar{x}) \leftrightarrows G_k \to 1$$

gives $G_k \to \mathrm{Aut}(\pi_1) \to \mathrm{Aut}(\pi_1^\ell)$

Prop (L-) $G_k \curvearrowright \mathbb{Q}_\ell[\![\pi_1^\ell]\!]/\mathcal{I}^n$ is semisimple.

<u>Pf</u> (1) Frob acts semi-simply on $H^1(X_{\bar{k}},\mathbb{Q}_\ell)$

(2) Trivial if H^1 is pure, as $\mathcal{I}$-adic filtration splits for wt reasons

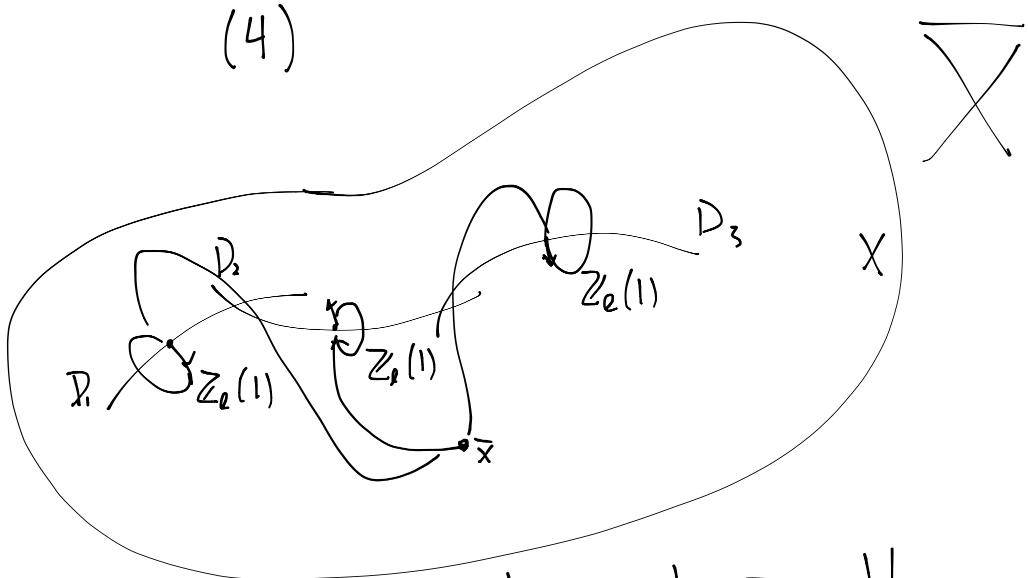
(3) In general, suffices to split G_K-equivariantly

$$0 \to \underbrace{\mathcal{I}^2/\mathcal{I}^n}_{\text{wts} \le 2} \to \mathcal{I}/\mathcal{I}^n \to \mathcal{I}/\mathcal{I}^2 = W^1\mathcal{I}/\mathcal{I}^2 \oplus W^2\mathcal{I}/\mathcal{I}^2 \to 0$$

wt spaces

- Splitting on W^1 exists for wt reasons

(4)



Idea: choose tangential basepoints $\bar{x}_i$ on bdry divisors D_i — inertia generates $W^2\mathcal{I}/\mathcal{I}^2$.

Let $\mathbb{Z}_\ell(1)\alpha_i \subset \pi_1^{\acute{e}t}(X_{\bar{k}}, \bar{x}_i)$ be inertia.

Then $\log \alpha_i \in \mathbb{Q}_\ell[[\pi_1^\ell(X_{\bar{k}}, \bar{x}_i)]]$ is a Frob e-vector of wt -2.

Want to move $\log \alpha_i$ to $\bar{x}$, G_K-equivariantly.

But $\exists!\, p_i \in \mathbb{Q}_\ell[[\pi_1^\ell(X_{\bar{k}}; \bar{x}, \bar{x}_i)]]$ s.t.

(1) p_i maps to 1 under augmentation

(2) p_i is G_k-fixed.

(Explain $\mathbb{Q}_\ell[[\pi_1^\ell(X_{\bar{k}}, \bar{x}, \bar{x}_i)]]$)

$p_i \log \alpha_i p_i^{-1}$ work as basis. $\square$

Hence $\mathcal{J}$-adic filtration splits G_k-equivariantly. (rationally)! But not integrally.

Convergent Gp Rings

$$\pi_n : \mathbb{Q}_\ell[[\pi_1^\ell]] \to \mathbb{Q}_\ell[[\pi_1^\ell]]/\mathcal{J}^n$$

$$v_n : \mathbb{Q}_\ell[[\pi_1^\ell]]/\mathcal{J}^n \to \mathbb{Z} \cup \{\infty\} \text{ obvious valuation.}$$

<u>Defn</u> (Convergent gp rings). Let $r \in \mathbb{R}_{>0}$.

Then $\mathbb{Q}_\ell[[\pi_1^\ell]]^{\leq \ell^{-r}} :=$

$$\{x \in \mathbb{Q}_\ell[[\pi_1^\ell]] \mid v_n(\pi_n(x)) + nr \to \infty\}$$

i.e. denominators grow linearly w/ slope r.

<u>Ex</u> $X = \mathbb{G}_m$, $\pi_1^\ell(X_{\bar{k}}, \bar{x}) = \mathbb{Z}_\ell$. Then

$$\mathbb{Q}_\ell[[\pi_1^\ell]] \simeq \mathbb{Q}_\ell[[T]], \quad \mathbb{Q}_\ell[[\pi_1^\ell]]^{\leq \ell^{-r}} \text{ is convergent}$$

$\gamma \mapsto 1+T$ power series of radius ℓ^{-r}.

Prop $\rho : \pi_1^\ell \xrightarrow[\text{cont.}]{} GL_n(\mathbb{Z}_\ell)$, triv mod ℓ^r.

Then if $r > r_0$, $\exists$ canonical diagram

$$\begin{array}{ccc} \mathbb{Z}_\ell[[\pi_1^\ell]] & \longrightarrow & \mathfrak{gl}_n(\mathbb{Z}_\ell) \\ \downarrow & & \downarrow \\ \mathbb{Q}_\ell[[\pi_1^\ell]]^{\leq \ell^{-r_0}} & \longrightarrow & \mathfrak{gl}_n(\mathbb{Q}_\ell) \end{array}$$

of cont. ring homomorphisms.

<u>Pf idea</u> triviality mod ℓ^{nr} beats denominator growth

<u>Cor</u> If $\mathbb{Q}_\ell[[\pi_1^\ell]]^{\leq \ell^{-r_0}}$ contains a set of Frob-eigen vectors w/ dense span, any ρ trivial mod ℓ is unip. Pure $\Rightarrow$ trivial.

<u>Pf</u> Weights.

<u>Thm 3</u> (L-) X sm. w/ snc compactification $\overline{X}$. Assume augmentation ideal $\mathcal{I} \subset \mathbb{Z}_\ell[[\pi_1^\ell]]$ satisfies

(*) $\mathcal{I}^n/\mathcal{I}^{n+1}$ torsion-free $\forall n$. Then $\exists$ explicit r_0 s.t. $\mathbb{Q}_\ell[[\pi_1^\ell]]^{\leq \ell^{-r_0}}$ contains a set of Frob e-vectors whose span is dense if

(1) $H^1(\overline{X}_{\bar{k}}, \mathbb{Q}_\ell) = 0$ (e.g. $\overline{X} = \mathbb{P}^1$), or

(2) $\dim H^1(X_{\bar{k}}, \mathbb{Q}_\ell) = 2$, ∞'ly many ℓ, or

(3) π_1^ℓ is nilpotent.

Implies Thms 1+2 by specialization argument.

Conjecture For any X sm. w/ snc cpctification + satisfying (*), $\exists$ such an r_0.

I.e. $\exists$ spanning $\{\gamma_i\} \subset \mathbb{Q}_\ell[\![\pi_1^\ell]\!]$ Frob. eigenvectors s.t. $v_n(\pi_n(\gamma_i)) = O(n)$

Thm 4 (L-) X as in the conjecture. There exist spanning set of Frob e-vectors $\{\gamma_i\}$ s.t. $v_n(\pi_n(\gamma_i)) = O(n \log n)$ for almost all ℓ. For remaining ℓ, $v_n(\pi_n(\gamma_i)) = O(n^2)$.

§ Estimating Denominators

Pf of thm 3(1). Gram-Schmidt

Want to keep track of denominators in splitting of

$$0 \to \mathcal{I}^2/\mathcal{I}^n \to \mathcal{I}/\mathcal{I}^n \to \mathcal{I}/\mathcal{I}^2 \to 0$$

Controlled by ℓ-adic distance between Frob. eigenvalues.

Ex

$$\begin{array}{ccccccccc} 0 \to & \mathbb{Z}_\ell & \to & \mathbb{Z}_\ell^2 & \to & \mathbb{Z}_\ell & \to 0 \\ & \circlearrowright 1 & & \circlearrowright \begin{pmatrix} 1 & 1 \\ 0 & 1+\ell \end{pmatrix} & & \circlearrowright \cdot (1+\ell) & \\ 0 \to & \mathbb{Z}_\ell & \to & \mathbb{Z}_\ell^2 & \to & \mathbb{Z}_\ell & \to 0 \end{array}$$

Doesn't split. Eigenvectors $\binom{1}{0}, \binom{1}{e}$.

<u>Pf of Thm 4</u> Yu's p-adic Baker thm

<u>Q</u> Bds are probably not optimal. (e.g. on polylogarithmic quotient). How to compute denominators. (ℓ-adic iterated integrals)

<u>Q</u> p-adic iterated integrals?