π_1 and Polylogarithms

§1) Motives, Realizations, and π_1

What is the category of mixed motives?

"Universal Cohomology Theory for Varieties."

Properties:

$$MM_{\mathbb{Q}} \xrightarrow{\pi_B} \text{Mixed Hodge Structures}$$

$$MM_{\mathbb{Q}} \xrightarrow{\pi_{dR}} \text{Bi-filtered } \mathbb{Q}\text{-vector spaces}$$

$$MM_{\mathbb{Q}} \xrightarrow{\pi_{\mathbb{Q}_\ell}} \ell\text{-adic Galois reps}$$

$$MM_{\mathbb{Q}} \xrightarrow{\pi_{cris}} \text{Filtered F-modules...}$$

Plus comparison theorems

$$\mathrm{comp}_{B,\mathbb{Q}_\ell} : \pi_B(M) \otimes_{\mathbb{Z}} \mathbb{Q}_\ell \xrightarrow{\sim} \pi_{\mathbb{Q}_\ell}(M)$$

$$\mathrm{comp}_{B,dR} : \pi_B(M) \otimes \mathbb{C} \xrightarrow{\sim} \pi_{dR}(M) \otimes \mathbb{C}$$

$$\mathrm{comp}_{cris,\mathbb{Q}_p} : \pi_{cris}(M) \otimes B_? \xrightarrow{\sim} \pi_{\mathbb{Q}_p}(M) \otimes B$$

$$\mathrm{comp}_{cris,dR} : \pi_{cris}(M) \xrightarrow{\sim} \pi_{dR}(M) \otimes \mathbb{Q}_p$$

preserving structures on both sides.

<u>Rem</u> Existence of $MM_{\mathbb{Q}}$ completely conjectural—

can work w/ "systems of realizations" or $MTM_{\mathbb{Q}}$ in some cases.

System of Realizations of $\pi_1(X)$ (X a genus 0 curve)

- $\pi_1^B(X_x) = \pi_1(X(\mathbb{C})^{an}, x)^{\text{pro-unip}}$
- $\pi_1^{dR}(X, x) = \mathrm{Aut}^{\otimes}(\omega_x)$ $\omega_x : \{\text{Unipotent v.b.'s on } X\} \to k\text{-Vector Spaces}$, $V \mapsto V_x$

Rem A unipotent v.b. is a v.b. w/ flat connection, which is an iterated extn of $(\mathcal{O}_X, d)$

- $\pi_1^{\mathbb{Q}_\ell}(X_x) = \pi_1^{et}(\bar{X}, x)^{(\ell)}$
- $\pi_1^{cris} = \cdots$

Prop $\pi_1^B(X)_{\mathbb{C}} \simeq \pi_1^{dR}(X)_{\mathbb{C}}$

"Pf." $\{\text{Unipotent v.b.'s on } X\} \xrightarrow{\sim} \{\text{Unipotent reps of } \pi_1(X(\mathbb{C})^{an}, x)\}$

$\wr\!\wr$

$\{\text{Unipotent reps of } \pi_1(X(\mathbb{C})^{an}, x)^{\text{pro-unip}}\}$

b) Explicit Descriptions of $\pi_1^B, \pi_1^{dR}, \pi_1^{\mathbb{Q}_\ell}$

(i) π_1^B

Pro-unipotent completion: G - discrete gp

$\mathbb{Q}[G]$ group ring, $\mathcal{J}_G \subset \mathbb{Q}[G]$ augmentation ideal.

$\widehat{\mathbb{Q}[G]} := \varprojlim_n \mathbb{Q}[G]/\mathcal{J}_G^n$.

Prop R-$\mathbb{Q}$-algebra. Then $G^{\text{pro-unip}}(R) = \{\text{gp-like elts in } \widehat{R[G]}\}$

Pf (1) unip reps of $G \xleftrightarrow{\sim}$ cts reps of $\widehat{R[G]}$

b/c I_G acts nilpotently $\Leftrightarrow$ G acts unipotently

(2) cts reps of $\widehat{R[G]} \xleftrightarrow{\sim}$ reps of gp-like elts in $\widehat{R[G]}$

(Hopf algebra lemma)

(ii) $\pi_1^{\mathbb{Q}_\ell}$

$\pi_1^{\mathbb{Q}_\ell} = (\pi_1^{B})^{(\ell)}$ = free pro-ℓ gp on $2g+n$ generators plus 1 relation

Galois action — very mysterious

(iii) π_1^{DR}

$\mathrm{Lie}\, \pi_1^{DR} = \widehat{\mathrm{Free\ Lie}}(H_1^{dR}(X))$ ← canonical grading

"Pf" (genus $X = 0$) $H^1_{dR}(X) = H^0(\bar{X}, \Omega^1(\log D))$

$H_1 = H^{1\vee}$

Let $\begin{matrix}\mathcal{E}\\ \downarrow \\ X\end{matrix}$ be a unipotent v.b. Then $H^1(X, \mathcal{O}_X) = 0 \Rightarrow \mathcal{E} \simeq \mathcal{O}^n$ as a coherent sheaf.

Connection extends uniquely to connection on $\mathcal{O}_{\bar{X}}^n$ w/ logarithmic connection at D, i.e. $\nabla = d + \omega$,

$\omega \in H^0(\bar{X}, \Omega^1(\log D)) \otimes \mathrm{End}\, V$, i.e. get

$\rho: H_1 \to \mathrm{End}\, V$ — obvs may recover ∇ from ρ.

Hence $(\mathcal{E}, \nabla)$ induces cts $\rho : \text{Free Lie}(H_1) \to \mathcal{E}_x$;
conversely, nilpotent rep admits filtration by trivials, hence induces unip. v.b. $\square$

§3) Comparison of $\mathbb{Q}$-structures on π_1^B, π_1^{dR}.

$X = \mathbb{P}^1 \setminus \{0, 1, \infty\}$

To compare rat'l structures, must take iterated integrals of $\omega \in (\text{Lie}\, \pi_1^{DR})^\vee$ along $\gamma \in \text{Lie}\, \pi_1^B$.

This is usually done on path torsors $\Pi(X, 0_{\vec{1}}, 1_{-\vec{1}})$,

get $$\int_0^1 \frac{dz_1}{z_1 - x_1} \int_0^{z_1} \frac{dz_2}{z_2 - x_2} \int_0^{z_2} \frac{dz_3}{z_3 - x_3} \cdots \int_0^{z_{n-1}} \frac{dz_n}{z_n - x_n}$$

$x_i \in \{0, 1\}$ (suitably regularized)

These integrals are special values of (multiple) polylogarithms, i.e. MZVs! (Pf expand in geometric series).

In general, comparing torsors $\Pi(X, 0_{\vec{1}}, a)$ get values of polylogarithms at a.

$$\text{Li}_{s_1, \ldots, s_k}(z) = \sum_{n_1 > n_2 > \cdots > n_k > 0} \prod_{j=1}^{k} n_j^{-s_j} \underline{z}^{n_j}$$

(Take s_i positive integers to get MZVs).

§4 Galois Action on π_1

$\pi_1(X_{\bar{\mathbb{Q}}})^{(\ell)}$ is nilpotent, torsion-free, hence has $\mathbb{Q}_\ell$-Lie algebra.

Let $\mathbb{Z}_\ell[[\pi_1]] = \varprojlim_{\pi_1 \to G} \mathbb{Z}_\ell[G]$, $\mathbb{Q}_\ell[[\pi_1]] = \varprojlim_n \mathbb{Q}_\ell[\pi_1]/\mathcal{I}_{\pi_1}^n$

$\mathrm{Lie}\,\pi_1$ = primitive elts in $\mathbb{Q}_\ell[[\pi_1]]$.

= free pro-nilpotent Lie algs on $2g+n$ generators mod $\underline{1}$ relation.

<u>Rem</u> $\mathcal{I}_G^n \cap \mathrm{Lie}\,\pi_1$ is lower central series filtration;

hence $gr.\,\mathrm{Lie}\,\pi_1 = gr.\,\pi_1(X_{\bar{k}})^{(\ell)} \otimes \mathbb{Q}_\ell$ as a Lie algebra.

Hence action on $gr.\,\mathrm{Lie}\,\pi_1$, or $gr\, \mathbb{Q}_\ell[[\pi_1]] = \bigoplus \mathcal{I}_G^n/\mathcal{I}_G^{n+1}$

totally determined by $G_k \curvearrowright \pi_1^{ab} = gr_0\, \mathrm{Lie}\,\pi_1 = \mathcal{I}/\mathcal{I}^2$.

Hence interesting question$/\mathbb{Q}_\ell$ is extra data.

<u>Analogy</u> $(\mathrm{Lie}\,\pi_1^{dR})_{\mathbb{C}}$ has canonical grading, but $\mathrm{Lie}\,\pi_1^B$ does not.

Hence choosing <u>any basis</u> of $(gr_0\, \mathrm{Lie}\,\pi_1^{dR})_{\mathbb{C}}$ get all periods up to finite dim'l change of basis matrix — i.e. periods detect canonical splitting of wt filtration on π_1^B + finite amount of data.

ℓ-adic polylogarithms compare $gr.\,\mathrm{Lie}\,\pi_1^{\mathbb{Q}_\ell}$, $\mathrm{Lie}\,\pi_1^{\mathbb{Q}_\ell}$.

<u>Better</u> understand Galois action on $\mathbb{Z}[[\pi_1^{(\ell)}]]$

<u>Defn</u> Choose free generators X_i of $\mathrm{Lie}\,\pi_1^{\mathbb{Q}_\ell}$. Then given $z \in X$, $\sigma \in \mathrm{Gal}(\bar{k}, k)$, w a word in the X_i, $w = i_1^{a_1} i_2^{a_2} \cdots$

$\mathrm{Li}^{\sigma}_{v}(z)$ = coeff of $[X_1, [X_{i_1} \cdots [X_{i_2} \cdots] \cdots]$ with $\underbrace{X_1, [X_{i_1}}_{\ell_1}$ and $\underbrace{X_{i_2} \cdots}_{\ell_2}$

in $\sigma(X_1) \in \mathrm{Lie}\,\Pi(X, O_T, z)$.

Rem The nilpotent qt of $\pi_1^{(\ell)}$ of class $\ell-1$ admits a $\mathbb{Z}_\ell$-Lie alg.
The p-adic valuations of these polylogarithms determine the G_n extn classes arising from $\Gamma_k \mathrm{Lie}\,\pi_1^{\{\ell\}}$ for k finite.

Rem In general, determine extn classes arising from

$$0 \to \mathcal{I}^2/\mathcal{I}^n \to \mathcal{I}/\mathcal{I}^n \to \mathcal{I}/\mathcal{I}^2 \to 0.$$

in $\mathbb{Z}_\ell[[\pi_1]]/\mathcal{I}^n$. (k finite).

Example $X = \mathbb{P}^1 \setminus \{0, 1, \infty\}$. $\iota : X \to \mathbb{G}_m$

Let $U = \ker \pi_1(\iota)$, $U^{ab} = \prod_{i \geq 1} \mathbb{Z}_\ell(i)$

Get exact sequence

$$0 \to U^{ab} \to \pi_1^{\mathrm{Lie}}(X)/[U, U] \to \mathbb{Z}_\ell(1) \to 0$$

Hence extn classes in $\mathrm{Ext}^1_{G_k}(\mathbb{Z}_\ell(1), \mathbb{Z}_\ell(i))$
for all $i \geq 1$.

Thm (Deligne) The order of this extn class is

$$-v_\ell\left(\tfrac{1}{2}\zeta(1-i)\right) \quad \text{(for } i \text{ even)}$$

and $k = \mathbb{Q}$.