Automorphisms of Blowups (joint w/ John Lesieutre)

§1) $X/\mathbb{C}$ a smooth (projective) variety

Goal: Describe $\underline{\mathrm{Aut}}(X)$, dynamics of $\mathrm{Aut}(X) \curvearrowright X$

Some open questions:

- Is $\pi_0(\underline{\mathrm{Aut}}(X))$ finitely gen'd? (open for rat'l surfaces!)
- $\varphi \in \mathrm{Aut}(X)$, $x \in X$, $Y \subset X$; describe $\{n \mid \varphi^n(x) \in Y\}$

Leave this on the board

(Dynamical Mordell-Lang; Bell, Ghioca, Tucker, Lagarias Poonen)

- Describe $\bigcup_n \varphi^n(Z) \cap Z$
- $Y, Z \subset X$ s.t. $\mathrm{codim}\, Y + \mathrm{codim}\, Z = \dim X$; $x \in Y \cap Z$ s.t. $\varphi(x) = x$. Bound $\{\mathrm{mult}_x\, \varphi^n(Y) \cap Z\}_n$ (Arnol'd)
- How to make examples? ($\kappa(X) \leq 0$)
  E.g. how does $\mathrm{Aut}(X)$ change after blowing up X? (Bayraktar-Cantat, Lesieutre, Truong)

§2) Examples + Statements

- $X = \mathbb{P}^2$, $\varphi \in PGL_2$ generic, $p \in \mathrm{Fix}(\varphi)$
  $L \subset \mathbb{P}^2$ a general line w/ $p \in L$.

Observations:

(a) $\bigcup_n \varphi^n(L) \cap L = \{p\}$

(b)
$$\begin{tikzcd} Bl_p X \arrow[r] \arrow[d, "\tilde{\varphi}"] \arrow[dr, phantom, "\circlearrowright"] & \mathbb{P}^1 \arrow[d, "\tau"] \\ Bl_p X \arrow[r] & \mathbb{P}^1 \end{tikzcd}$$
$\tilde{\varphi}^n(L)$ all disjoint

(c) $H^0(L, N_{L/X}) \neq 0$

(d) $L \sim_{rat}$ periodic divisor

We show (a) $\Rightarrow$ (b), (c), (d) in general.

<u>Thm</u> (Lesieutre, L-) X, sm. proj., $D \subset X$ divisor, $\varphi : X \dashrightarrow X$.

Suppose $\overline{\bigcup_n \varphi^n(D)} \cap D \neq D$. Then after replacing φ w/ φ^N, $\exists$ birat'l morphism $Y \to X$ w/ Y sm. s.t.

(1) φ lifts to $\tilde{\varphi} : Y \to Y$

(2) $\tilde{\varphi}^n(\tilde{D})$ are disjoint

(3) $\exists$
$$\begin{tikzcd} Y \arrow[r, "\sim", "\tilde{\varphi}"'] \arrow[d, "f"'] \arrow[dr, phantom, "\circlearrowright"] & Y \arrow[d, "f"] \\ C \arrow[r, "\sim", "\tau"'] & C \end{tikzcd}$$

(4) $H^0(D, N_{D/X}) \neq 0$.

(5) If $\bigcup \varphi^n(D) \cap D \neq \emptyset$ or $\pi_1(X) \neq 1$, $D \sim_{rat}$ periodic divisor.

- C - genus 2 curve s.t. $A = Jac(C)$ is CM by K

K3 surface $X = Bl_{A[2]} A/\{\pm 1\}$, $Y = \widetilde{C/\{\pm 1\}} \simeq \mathbb{P}^1 \subset X$

$\varphi \in \mathcal{O}_K^\times$ of infinite order.

(a) $\bigcup_n \varphi^n(Y) \cap Y$ is dense in Y

(b) $H^0(Y, N_{Y/X}) = \{0\}$

We show: (b) $\Rightarrow$ (c) in general

- $C_1, C_2 \subset \mathbb{P}^2$ gen'l smooth cubics, $X = \mathrm{Bl}_{C_1 \cap C_2} \mathbb{P}^2$.

$s_i \left(\begin{array}{c} X \\ \pi \downarrow \\ \mathbb{P}^1 \end{array} \right.$ ell. fibration, $\{s_i\}_{i=1,\dots,9}$ sections. $\rightsquigarrow \mathbb{Z}^7 \curvearrowright X$

$\mathrm{Aut}(\mathbb{P}^2) = PGL_3$, $\mathrm{Aut}(X) \supset \mathbb{Z}^7, \dots$ Can we modify Aut via blowups in codim ≥ 2?

We show: $\dim Y \ll \dim X$; then $|\pi_0(\mathrm{Aut}\, X)| < \infty \Rightarrow |\pi_0(\mathrm{Aut}\, \mathrm{Bl}_Y X)| < \infty$.

Thm (Lesieutre, L-) X sm. proj. $/\mathbb{C}$, sm. $Y \subset X$. Suppose

(*) (1) $\dim X \leq 4$, $\dim Y \leq \dim X - 3$, or
(2) $2 \dim Y + 3 \leq \dim X$.

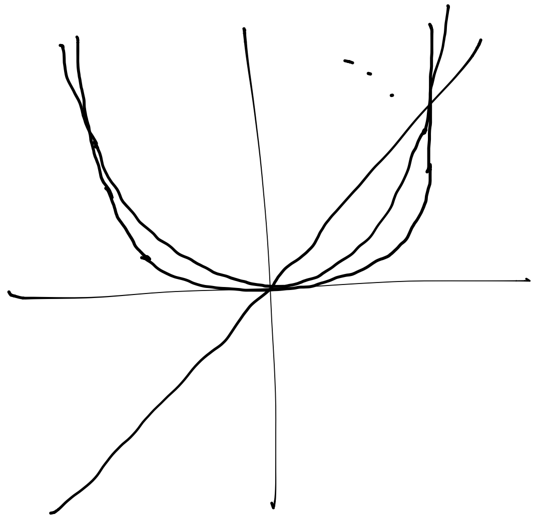
Then $\exists N \in \mathbb{Z}_{>0}$ s.t. $\forall \varphi \in \mathrm{Aut}(\mathrm{Bl}_Y X)$, φ^N descends to X.

Cor X, Y as above. Then $|\pi_0(\mathrm{Aut}\, X)| < \infty \Rightarrow |\pi_0(\mathrm{Aut}\, \mathrm{Bl}_Y X)| < \infty$.

Rem. - rk Pic $X = 1$: Bayraktar - Cantat
- Various hypotheses on $\overline{NE}(X)$: Truong

Q Can one replace (*) w/ $\dim Y \leq \dim X - 3$?

- $X = \mathbb{A}^2$, $Y \subset X$, $Y = \{(x,y) \mid y = x\}$, $Z = \{(x,y) \mid y = 0\}$
$\varphi : \mathbb{A}^2 \to \mathbb{A}^2$, $(x,y) \mapsto (x^2, y)$



- $\mathrm{mult}_0\, \varphi^n(Y)\cap Z \to \infty$
- Cannot separate $\varphi^n(Y)$ by blowups.

<u>Thm</u> (Lesieutre, L-) $\varphi : X \xrightarrow{\sim} X$, $V \subset X$ w/ $\varphi(V) = V$.
$Y, Z \subset X$ Cohen-Macaulay w/ $V \subset Y, Z$,
$$\mathrm{codim}_X Y + \mathrm{codim}_X Z = \mathrm{codim}_X V.$$
Then $\{\mathrm{mult}_V\, \varphi^n(Y)\cap Z \mid \dim_V \varphi^n(Y)\cap Z = \dim V\}$
is bdd.

<u>Rem</u> Case V is a pt is a theorem of Arnol'd.

§3) Dynamical Mordell-Lang

All of the above thms follow from a new version of the Dynamical Mordell-Lang Conj., proved via p-adic analysis.

<u>Defn</u> $A \subset \mathbb{Z}$ is semi-linear if it is the union of a finite set and a set of residue classes mod N. (Say it has length dividing N)

<u>Thm</u> (Lesieutre, L-) X sm., $\varphi \in \mathrm{Aut}(X)$ $Y, Z \subset X$, $\{Y_k\}_{k\in\mathbb{N}}$ a sequence of subschemes supported on Y w/ finitely many associated pts, + "of finite type." Then

$$A_k = \{ n \mid \varphi^n(Y_k) \subset Z \}$$

is semilinear of length N, N independent of k.

Ex $Y_k = V(\mathcal{J}_Y^k) \times_X W$ for some fixed $W \subset X$.

Verbally define "finite type."

Pf "p-adic methods"

$X = \mathbb{A}^n_{\mathbb{Z}}$, $Y = p \in \mathbb{A}^n$, $Z = \{x_1 = 0\}$, $\varphi \in GL_n(\mathbb{Z})$

Rem This is the Mahler-Skolem-Lech theorem

$\varphi^N = \mathrm{id} \bmod 3$

Consider $f: \mathbb{Z}_3 \to GL_n(\mathbb{Z}_3)$

$z \mapsto \exp(z \log \varphi)$

Let $A_k = \{ z \mid f(z) \cdot \varphi^k(p) \}$ $\quad 0 \leq k < N$.

Set of zeroes of p-adic analytic function, hence finite or all of $\mathbb{Z}_3$.

Real idea: Work in p-adic polydisc using interpolation result of Poonen, reduce to local situation. $\boxed{Z}$

Rem False in char $p > 0$

§4) Applications

(i) Separating iterates

Ex. X - surface, $\varphi \in \mathrm{Aut}\, X$; $x \in \mathrm{Fix}(\varphi)$

$Y, Z \subset X$ curves w/ $x \in Y, Z$

Let $Y^{(k)}$ = k-th order germ of Y at x.

Then $A(Y^{(k)}, Z) = \{ n \mid \varphi^n(Y) \text{ tangent to } Z \text{ to order } k \}$

Obstacles to separating $Y, \varphi^n(C), \ldots$ by blowups.

(1) Suppose Y tangent to $\varphi^n(Y)$ when n perfect square

$\tilde{\varphi}: Bl_x X \to Bl_x X$ does not lift after blowing up at x', even after $\varphi \rightsquigarrow \varphi^N$

Need $\varphi^n(Y)$ tangent $\forall n$ or for no n.

(2) Y tangent to $\varphi^{2^k n}(C)$ to order k.

$\Rightarrow$ then no blowup can separate $\varphi^{2^k n}(C)$

(consistent w/ semilinearity but not uniform bd)

Rem. Also gives bd on $\mathrm{mult}_x(\varphi^n(Y) \cap Z)$

b/c $A(Y^{(k)}, Z) \supset A(Y^{(k+1)}, Z) \supset \cdots$

w/ uniform bd.

(ii) Automorphisms of Blowups

Lemma Y sm, $\pi_i: Y \to X_i$ realizing Y as $Bl\, X_i$ along sm. centers of dim $\leq r$.

(1) If $2r+3 \leq \dim Y$, exceptional divisors are equal or disjoint

(2) If $\dim Y \leq 4$, $r \leq \dim Y - 3$, $\exists$ irred $W \subset E_0$ s.t. $E_0 \cap E_i \subset W$ $\forall i > 0$.

Pf (1) Fibers of $E_i \to X_i$ must intersect for dimension reasons along subvar of dim > 0

(2) E_0 must contain a subvar. contracted by π_j.

(i) $E_0 = \mathbb{P}^3$ ✓

(ii) $E_0 \to C$ a $\mathbb{P}^2$-bdl — $\mathrm{rk}\, NS\, E_0 = 2$, hence at most 2 contractions.

Cor $\pi: Y \to X$ blowup along sm. centers w/ codim as above.
Then $\varphi^N(E) = E$ for some N.

Pf (1) By lemma applied to $\pi \circ \varphi^n$, exceptionals permuted.
(2) $\bigcup_n \varphi^n(E) \cap E$ Zar. closed. But $H^0(E, N_{E/X})$
$= 0$

Cor Uniform bd

Cor Finiteness of Aut(B(.))
Pf Lieberman + ε.

Pf Mahler-Skolem-Lech on $NS(X)$.

Q X w/ $\mathrm{Aut}(X) = \{1\}$. Can one blow up in codim ≥ 3 and get Y w/ $\mathrm{Aut}(Y)$ infinite?

Rem No for $\dim X \leq 4$.