Non-Abelian Lefschetz Thms

§1) Classical Lefschetz Thms

X - sm. proj. var$/\mathbb{C}$

$D \subset X$ ample divisor

$\{\text{Properties of } D\} \longleftrightarrow \{\text{Properties of } X\}$

<u>Thm</u>

$\pi_1(D) \xrightarrow{\sim} \pi_1(X)$ <u>if</u> $\dim X \geq 3$

$\mathrm{Pic}\, X \xrightarrow{\sim} \mathrm{Pic}\, D$ <u>if</u> $\dim X \geq 4$

$H^i(X, A) \to H^i(D, A)$ <u>if</u> $i < \dim D$

<u>Reinterpretation</u>

$\underline{\mathrm{Hom}}(X, BG) \xrightarrow[\text{equiv}]{\sim} \underline{\mathrm{Hom}}(D, BG)$

$\underline{\mathrm{Hom}}(X, B\mathbb{G}_m) \xrightarrow[\text{equiv}]{\sim} \underline{\mathrm{Hom}}(D, B\mathbb{G}_m)$

$\mathrm{Hom}_{\mathrm{cont}}(X, K(A,i)) \xrightarrow[\text{wk equiv}]{\sim} \underline{\mathrm{Hom}}_{\mathrm{cont}}(D, K(A,i))$

<u>Ex</u> Cohomology of hypersurface

<u>Rem</u> Make sense of $K(A,i)$ in AG.

<u>Idea</u> Y a space (scheme, stack, etc.)

Study $\underline{\mathrm{Hom}}(X, Y) \xrightarrow{\text{res}} \underline{\mathrm{Hom}}(D, Y)$

- When is it isom? mono? ...
- Which maps $D \to Y$ extend to maps $X \to Y$?

<u>Slogan</u> Cohomology w/ coeffs in Y

<u>Rem</u> Generalization, taking $Z = X \times Y$, $f = pr_X$.

Idea
$$\begin{tikzcd} Z_D \arrow[r] \arrow[d, "f_D"'] \arrow[dr, phantom, "\square"] & Z \arrow[d, "f"] \\ D \arrow[r, hook] & X \end{tikzcd}$$
Study $\underline{\text{Sections}(f)} \to \text{Sections}(f_D)$

Slogan Cohomology w/ coeffs in local system Z.

Rem Esp. interesting when Y represents a "moduli functor."

§2) Main Results + Applications (char $k=0$)

char $p>0$ versions later

Thm 1 (Sommese, L-) Y DM stack w/ q-proj. coarse space, $f: D \to Y$ s.t. $\dim \operatorname{im}(f) < \dim D - 1$. Then f extends uniquely to $\tilde{f}: X \to Y$. Rem $\dim X \geq 3$.

Rem Special cases due to Sommese.

Defn A v.b. $\mathcal{E}$ is ample (resp. nef) if $\mathcal{O}(1)$ on $\mathbb{P}(\mathcal{E})$ is ample (resp. nef).

Rem Slightly weaker defn of nef: $\forall C \xrightarrow{f} X$, and $f^*\mathcal{E} \to \mathcal{L} \to 0$, one has $\deg_C \mathcal{L} \geq 0$.

(Doesn't require base to be proper).

Thm 2 (L-) Y sm, $\dim Y < \dim D$, $f: D \to Y$

$\dim X \geq 3$ $\quad N_{D/X} \otimes f^*\Omega^1_Y$ ample.

Then f extends uniquely to $\tilde{f}: X \to Y$.

<u>Thm 3</u> (L-) $\begin{array}{ccc} Z_D & \to & Z \\ g_D \downarrow \uparrow & \square & \downarrow \uparrow g \\ D & \to & X \end{array}$ sm. proper, $\dim X \geq 3$, rel $\dim g < \dim D$,

$\begin{array}{c} Z_D \\ g_D \downarrow \uparrow f \\ D \end{array}$ s.t. $N_{D/X} \otimes f^* \Omega^1_{Z/X}$ ample.

Then f extends uniquely to section $\tilde{f} : X \to Z$.

<u>Cor</u> (2') Ω^1_Y nef $\Rightarrow$ $\mathrm{Hom}(X, Y) \to \mathrm{Hom}(D, Y)$ isom.

<u>Cor</u> (3') $\Omega^1_{Z/X}$ nef $\Rightarrow$ Sections$(g) \to$ Sections(g_D) isom.

<u>Rem</u> This is main application.

<u>Ex</u> (Varieties w/ nef cotangent bdl)

(1) Curves of genus ≥ 1

(2) Abelian varieties

(3) $\mathrm{Sym}^n \Omega^1$ glob. gen'd

(4) Cpct qts of bdd domains in $\mathbb{C}^n$

(5) $\mathcal{M}_{1,n}$ $(n \geq 1)$, $\mathcal{M}_{g,n}$, $g \geq 2$. $\mathcal{A}_g$, moduli of hyperkählers, moduli of CYs, Shimura varieties

Clearly state corollary!

<u>Ex</u> (Maps $Z \to X$ w/ $\Omega^1_{Z/X}$ nef)

(1) $\Omega^1_{Z/X}$ relatively glob. gen'd (e.g. families of curves of genus ≥ 1, (torsors for) Abelian schemes)

(2) Ω^1_Z nef $\Rightarrow$ $\Omega^1_{Z/X}$ nef.

(These results have an analytic nature, unlike pf of main thm) (Griffiths semi-positivity)

Rem. Logarithmic version

Rem. All results pass through char $p>0$.

§3) The Lefschetz Package (SGA 2)

$$
\begin{tikzcd}
D \arrow[d] & & \mathrm{Hom}(D,Y) \\
\hat{D}\text{ - formal scheme} \arrow[d] & & \mathrm{Hom}(\hat{D},Y) \arrow[u, "\text{deformation theory}"'] \\
U\text{ - Zariski-open containing } D \arrow[d] & \rightsquigarrow & \varinjlim \mathrm{Hom}(U,Y) \arrow[u, "\text{algebraization}"'] \\
X & & \mathrm{Hom}(X,Y) \arrow[u, "\text{extension}"']
\end{tikzcd}
$$

Want to show that each of these maps is an isomorphism.

Ex. Lefschetz for Pic (SGA 2)

$$
\begin{tikzcd}
\mathrm{Pic}\, D \\
\mathrm{Pic}\, \hat{D} \arrow[u] \\
\mathrm{Pic}\, U \arrow[u] \\
\mathrm{Pic}\, X \arrow[u]
\end{tikzcd}
$$

$\mathrm{Pic}\,\hat{D} \to \mathrm{Pic}\, D$: $1 \to 1+\mathcal{I}_D^n/\mathcal{I}_D^{n+1} \to \mathcal{O}^*_{D_{n+1}} \to \mathcal{O}^*_{D_n} \to 1$ + Kodaira vanishing

$\mathrm{Pic}\, U \to \mathrm{Pic}\,\hat{D}$: Left: compare $H^i(X,\mathcal{E}) \to H^i(D,\mathcal{E}|_D)$ then algebraize resol'n (SGA2, Exposé XII)

$\mathrm{Pic}\, X \to \mathrm{Pic}\, U$: Hartog's thm.

Thm (L-, Extension + Algebraization)

X sm. proj.; $D \subset X$ ample, $\hat{D}$ associated formal scheme, $\dim X \geq 3$. Then:

$$\mathrm{Hom}(X,Y) \to \mathrm{Hom}(\hat{D},Y)$$

is an isomorphism if:

(*)
(1) $\dim Y < \dim D$, Y normal, or
(2) Y normal proper, contains no rat'l curves.
(3) ...

Rem. Prove (2) by resolving, contracting rat'l curves.

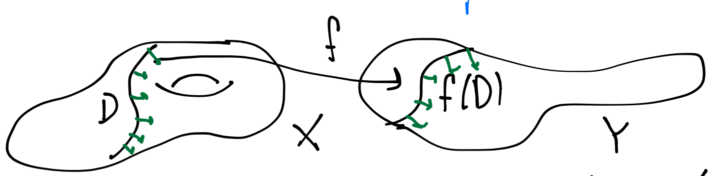
Rem (1) Hypotheses can be weakened.

(2) Follows from algebraization result for perfect complexes.

Reduces question to pure deformation theory.

Rem Prove your own Lefschetz thms!

§4) Deformation Theory



Deformations torsor for $\mathrm{Hom}(N_{D/X}, f^*T_Y)$

Obstructions lie in $\mathrm{Ext}^1(N_{D/X}, f^*T_Y)$

Want both groups to vanish so deformations exist and are unique.

(I.e. positivity for $N_{D/X} \otimes f^*\Omega^1_Y$.)

Rem If D sm. + $N_{D/X} \otimes f^*\Omega^1_Y$ ample, $\dim Y < \dim D$, can use le Potier vanishing. In general, need char $p>0$ techniques.

Rem Will use techniques due to Arapura/Bhatt-de Jong (pushing Deligne-Illusie)

Defn (f-amplitude) k-field, X k-scheme, $\mathcal{E}$ a v.b. on X.

(char $k = p > 0$) $\phi(\mathcal{E}) = \min\{ i_0 \mid H^i(X, \mathcal{F} \otimes \mathcal{E}^{(p^k)}) = 0$ for $i > i_0$, $k \gg 0$, all coh. $\mathcal{F} \}$

(char $k = 0$) $\phi(\mathcal{E})$: Choose model of $(X, \mathcal{E})$ over f.t. $\mathbb{Z}$-scheme S s.t. $\max_{q \in S^{cl}} \phi(\mathcal{E}_q)$ is minimal. This is $\phi(\mathcal{E})$.

<u>Thm</u> (char $k = p > 0$) X sm proj, $D \subset X$ ample, Y smooth. $f: D \to Y$ morphism s.t.

$\dim X \geq 3$, and $\phi(N_{D/X} \otimes f^* \Omega^1_Y) < \dim D - 1$.

Then $F^n_{Y/k} \circ f : D \to Y^{(p^n)}$ extends to $\hat{D}$ for $n \gg 0$.

<u>Pf</u> (Assume $k = \mathbb{F}_p$, so $F = F_{-/k}$)

$$\begin{array}{ccc} D_{p^n} & & \\ \uparrow & & \\ D & \xrightarrow{F^n} & D \\ f\downarrow & & \downarrow f \\ Y & \xrightarrow{F^n} & Y \end{array}$$

Extend $f \circ r$ to D_{p^n+1}: $\mathrm{Ext}^i(r^* f^* \Omega^1_Y, \mathcal{I}_D^{p^n}/\mathcal{I}_D^{p^n+1})$ $\quad i = 0,1$

$= H^i(D, N_{D/X}^{p^n \vee} \otimes F^{n*} f^* T_Y)$

$= H^{\dim D - i}(D, K(D) \otimes F^{n*}(N_{D/X} \otimes f^* \Omega^1_Y))^\vee$

$= 0$

$\blacksquare$

Now: when can we "factor" an extension

$$\begin{array}{ccc} D & \xrightarrow{F^n \circ f} & Y \\ f \downarrow & & \\ X & \tilde{f} & \end{array}$$

through Frob?

Thm X sm proj$/k$, $D\subset X$ ample, Y smooth, satisfies $(*)$. $f: D\to Y$ morphism s.t.
$\dim X\geq 3$ and $\phi(N_{D/X}\otimes f^*\Omega^1_Y)<\dim D-1$.
If char $k=0$ or char $k=p>\dim X$ and X lifts to $W_2(k)$ (k perfect)
then f extends to a map $X\to Y$.
If $\phi(N_{D/X}\otimes f^*\Omega^1_Y)<\dim D$, extns are unique.

Rem This condition can be checked. For example,
(Arapura) $\phi(\mathcal{E})<\operatorname{rk}\mathcal{E}$ for $\mathcal{E}$ ample (in char 0)
(Pre-Thm. True in char $p>0$.)
(Follows from Ω^1_Y nef — typically check analytically.)

Pf of thm $\tilde{f}: X\to Y$ factors through Frob $\Leftrightarrow$ $\tilde{f}^*\Omega^1_Y\to\Omega^1_X$ is zero.
$\tilde{f}^*\Omega^1_Y\to\Omega^1_X\to\Omega^1_D$ is zero, so get map $\tilde{f}^*\Omega^1_Y\to N^\vee_{D/X}$
— zero by Deligne-Illusie type argument

$$\begin{cases} 0\to N^\vee_{D/X}\to\Omega^1_X|_D\to\Omega^1_D\to 0\\ 0\to\Omega^1_X(-D)\to\Omega^1_X\to\Omega^1_X|_D\to 0\end{cases}$$

hence get map $\tilde{f}^*\Omega^1_Y\to\Omega^1_X(-D)$
— zero by DI-type argument. $\blacksquare$

Rem f-amplitude designed for this type of argument.

Defn k-field, X-k-scheme, $\mathcal{E}$ a v.b. on X, $\mathcal{O}(1)$ an ample l.b.
(char $k=p>0$): $\mathcal{E}$ is f-semipositive if $\exists n>0$ s.t.

$$H^i(X, \mathcal{E}^{(p^k)}(n))=0 \text{ for all } k\in\mathbb{N}, \text{ all } i>0.$$

(char $k=0$): $\mathcal{E}$ f-semipositive if $\exists$ model of $(X, \mathcal{E}, \mathcal{O}(1))$ over finite-type $\mathbb{Z}$-scheme whose special fibers are.

Thm X, D as before ($\dim X\geq 3$), and Y sm. w/ Ω^1_Y f-semipositive. Then

$$\mathrm{Hom}(X,Y)\to \mathrm{Hom}(D,Y) \text{ is isom.}$$

Ex Families of curves of genus ≥ 1 or AVs over curves of genus ≥ 1, free gp qts of such, etc. have Ω^1 f-semipositive.

Rmk (1) All these thms have analogues for

$$\mathrm{Sections}(f)\to \mathrm{Sections}(f_D)$$

(2) Applications to VHS (horizontal cot. bdle nef).

(3) Y singular (e.g. many moduli spaces): What are positivity properties of L_Y?

Use this to prove your own Lefschetz Thms!

Other Applications

- Classifying varieties w/ special divisors

Conj (Sommese) X sm. proj. variety, $Y \subset X$ ample divisor, $p: Y \to Z$ a $\mathbb{P}^d$-bundle. Then either

(1) $X = \mathbb{P}^3$, $Y = \mathbb{P}^1 \times \mathbb{P}^1$ a quadric.

(2) $X = Q \subset \mathbb{P}^4$ a quadric, Y hyperplane section, $Z = \mathbb{P}^1$

(3) $Y = \mathbb{P}^1 \times \mathbb{P}^{n-2}$, $Z = \mathbb{P}^{n-2}$, $X = \mathbb{P}(\mathcal{E})$ for $\mathcal{E}$ ample on $\mathbb{P}^1$, or

(4) $X = \mathbb{P}(\mathcal{E})$, $\mathcal{E}$ ample v.b. on Z, $\mathcal{O}(Y) = \mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$.

Rem • Sommese: true for $d > 1$

• Beltrametti–Ionescu: true for $\dim Z \leq 2$

...

Content: Either $p: Y \to Z$ extends to $\tilde{p}: X \to Z$, or $Z = \mathbb{P}^n$.

Reduces to:

Conj X conn'd sm. proj., $\mathcal{E}$ ample v.b. on X, $\mathrm{Hom}(\mathcal{E}, T_X) \neq 0$. Then $X = \mathbb{P}^n$.