A pairing on Br(X)'

Goal: Prove Thm 5.1 of Tate —

<u>Thm</u> k-finite field, $X/_k$ sm. proj. surface. Then there is a canonical skew-symmetric pairing

$$Br(X)[\ell^\infty] \times Br(X)[\ell^\infty] \to \mathbb{Q}_\ell/\mathbb{Z}_\ell$$

if $\ell \nmid \# k$. The kernel is exactly the divisible elts.

<u>Rem</u> True for $\ell = \mathrm{char}\, k (=p)$ also — see Thm 2.4 in Milne. Thus obtain skew-symmetric pairing

$$Br(X) \times Br(X) \to \mathbb{Q}/\mathbb{Z}$$

whose kernel is exactly the divisible elts.

<u>Cor</u> $\#|Br(X)|$ is a square or twice a square if finite (we prove this for the prime-to-p part of $\#|Br(X)|$)

Quick Review of Étale Cohomology (of torsion sheaves, $\mathbb{G}_m$)

<u>Recall</u>: Étale topology defined by using "jointly surjective étale maps $\{U_i \to X\}$ instead of Zariski covers $\{U \hookrightarrow X\}$ in all defns. $X_{ét}$ is category of all étale X-schemes.

<u>Defn</u> A functor $F: X_{ét}^{op} \to C$ is an étale sheaf if for $\{U_i \to Y\}$ an étale cover of étale X-schemes,

replace $\cap$ w/ fiber product

$$F(Y) \to \prod_i F(U_i) \rightrightarrows \prod_{i,j} F(U_i \times_Y U_j) \text{ is exact.}$$

Ex/Thm. Representable functors are sheaves (étale descent of morphisms).

1) $\mathbb{G}_m : U \mapsto \Gamma(U, \mathcal{O}_U^*)$

2) $\mu_n : U \mapsto \{ f \in \Gamma(U, \mathcal{O}_U^*) \mid f^n = 1 \}$

3) Quasi-coherent sheaves on X, $\tilde{\mathcal{F}}$

$$(U \xrightarrow{f} X) \longmapsto \Gamma(U, f^*\mathcal{F})$$

Can define $H^i(X, \tilde{\mathcal{F}})$ using usual derived functor business or (if X quasi-projective) Čech cohomology.

$$\prod \tilde{\mathcal{F}}(U_i) \rightrightarrows \prod_{i,j} \mathcal{F}(U_i \times_X U_j) \rightrightarrows \prod_{i,j,k} \mathcal{F}(U_i \times_X U_j \times_X U_k) \rightrightarrows \cdots$$

(descent of line bdls)

Facts $H^1(X, \mathbb{G}_m) = H^1(X_{\text{ét}}, \mathbb{G}_m)$ (Hilbert 90)

$H^i(X, \tilde{\mathcal{F}}) = H^i(X_{\text{ét}}, \tilde{\mathcal{F}})$ "additive Hilbert 90" aka descent of q.c. sheaves + morphisms...

However $H^1(X, PGL_n) \neq H^1(X_{\text{ét}}, PGL_n)$

$H^2(X, \mathbb{G}_m) \neq H^2(X_{\text{ét}}, \mathbb{G}_m)$

Ex $H^1(\operatorname{Spec} \mathbb{R}, PGL_2) = ?$ Ex k = field, $H^i((\operatorname{Spec} k)_{\text{ét}}, \mathcal{F})$ = Galois cohom.

Defn $\operatorname{Br}(X) = H^2(X_{\text{ét}}, \mathbb{G}_m)_{\text{tors}}$ ("cohomological Brauer group")

Rem Usually defined as $\bigcup_n \mathrm{im}(H^1(X_{\text{ét}}, PGL_n) \to H^2(X_{\text{ét}}, \mathbb{G}_m))$
(Azumaya algebras $/\sim$, where $A_1 \sim A_2$ if $A_1 \otimes A_2^{op} = \mathrm{End}(\mathcal{E})$)
A - étale locally $\simeq \mathrm{End}(\mathcal{O}^n)$

Kummer Sequence + Corollaries

Prop $1 \to \mu_n \to \mathbb{G}_m \xrightarrow{f \mapsto f^n} \mathbb{G}_m \to 1$ is exact sequence of étale sheaves on $X_{\text{ét}}$ if n is invertible on X.

Pf Must show $\mathbb{G}_m \xrightarrow{f \mapsto f^n} \mathbb{G}_m$ is surjective. I.e. if $f \in \Gamma(U, \mathcal{O}_U^*)$ is an invtble fn, $\exists V \xrightarrow{s} U$ étale and $g \in \Gamma(V, \mathcal{O}_V^*)$ w/ $g^n = s^* f$.

$$\begin{tikzcd} V \arrow[r, "g"] \arrow[d] \arrow[dr, phantom, "\square"] & \mathbb{G}_m \arrow[d, "t \mapsto t^n"] \\ U \arrow[r, "f"] & \mathbb{G}_m \end{tikzcd}$$

étale surjection hence $V \to U$ is étale surjection. $\boxed{\Sigma}$

Cor LES

$$\cdots \to H^{i-1}(X_{\text{ét}}, \mathbb{G}_m) \to H^i(X, \mu_n) \to H^i(X, \mathbb{G}_m) \xrightarrow{\cdot n} H^i(X, \mathbb{G}_m) \to \cdots$$

In low degrees:

$$1 \to \mu_n(X) \to \mathbb{G}_m(X) \xrightarrow{\cdot n} \mathbb{G}_m(X) \to H^1(X, \mu_n) \to \mathrm{Pic}\, X \xrightarrow{\cdot n} \mathrm{Pic}\, X \to H^2(X, \mu_n) \to$$
$$\mathrm{Br}(X) \xrightarrow{\cdot n} \mathrm{Br}(X) \to \cdots$$

Hence get:

$$
\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
 & 0 \arrow[d] & \mathrm{Pic}\, X/m\, \mathrm{Pic}\, X \arrow[r] \arrow[d] & (NS(\bar{X})/m\, NS(\bar{X}))^G \arrow[d] & \\
0 \arrow[r] & H^1(\bar{X}, \mu_m)_G \arrow[r] \arrow[d] & H^2(X_{\acute{e}t}, \mu_m) \arrow[r] \arrow[d] & H^2(\bar{X}, \mu_m)^G \arrow[r] \arrow[d] & 0 \\
 & (\mathrm{Pic}\, \bar{X})[m]_G \arrow[d] & \mathrm{Br}(X)[m] \arrow[r] \arrow[d] & \mathrm{Br}(\bar{X})[m]^G \arrow[r] & 0 \\
 & 0 & 0 & &
\end{tikzcd}
$$

(Diagram 5.1 in Tate)

Want to add red part of diagram, where $G := \mathrm{Gal}(\bar{k}/k) \simeq \hat{\mathbb{Z}} \ni \sigma$ Frob

and M^G, M_G is invts/coinvts, resp., i.e.

$M^G := \ker(1-\sigma : M \to M)$

$M_G := \mathrm{coker}(1-\sigma : M \to M)$

Galois cohomology for k

Recall: $G = \mathrm{Gal}(\bar{k}/k) = \hat{\mathbb{Z}}$, A a discrete G-module

Prop (1) $H^0(G, A) = A^G$

(2) $H^1(G, A) = A_G$

(3) $H^i(G, A) = 0$ for $i \geq 2$, A torsion or divisible

Pf Exercise, or Neukirch

Cor X a k-variety, $\bar{X} = X_{\bar{k}}$. Then there are natural SES's

$$0 \to H^{i-1}(\bar{X}, A)_G \to H^i(X, A) \to H^i(\bar{X}, A)^G \to 0$$

for A torsion sheaf on $X_{\acute{e}t}$.

Pf Hochschild-Serre SS $E_2^{p,q} : H^p(k, H^q(\bar{X}, A)) \Rightarrow H^{p+q}(X, A)$,

degenerates as coh. dim$(k) = 1$.

Thus obtain

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d, "\text{b/c } (M \mapsto M^G) \text{ is left-exact}"] & \\
 & & \operatorname{Pic} X / m \operatorname{Pic} X \arrow[r] \arrow[d] & (\operatorname{Pic} \bar{X} / m \operatorname{Pic} \bar{X})^G \arrow[d] & \\
0 \arrow[r] & H^1(\bar{X}, \mu_m)_G \arrow[r] \arrow[d] & H^2(X, \mu_m) \arrow[r] \arrow[d] & H^2(\bar{X}, \mu_m)^G \arrow[r] \arrow[d] & 0 \\
 & \underbrace{\operatorname{Pic} \bar{X}[m]_G}_{\text{Kummer} + -_G} & \operatorname{Br}(X)[m] \arrow[r] \arrow[d] & \underbrace{\operatorname{Br}(\bar{X})[m]^G}_{\text{Kummer} + -^G} & \\
 & & 0 & &
\end{tikzcd}$$

Claim $H^1(X, \mu_m)_G \to \operatorname{Pic} \bar{X}[m]_G$ is an isomorphism.

Pf $0 \to H^0(\bar{X}, \mathbb{G}_m) \xrightarrow{m} H^0(\bar{X}, \mathbb{G}_m) \to H^1(\bar{X}, \mu_m) \to \operatorname{Pic} \bar{X}[m] \to 0$

surjective b/c X proper, $\bar{k}$ alg. closed, hence $H^1(\bar{X}, \mu_m) \to \operatorname{Pic} \bar{X}[m]$.

$+ \; -_G$ is a functor.

Claim $\operatorname{Pic} \bar{X} / m \operatorname{Pic} \bar{X} \cong NS(\bar{X}) / m NS(\bar{X})$.

Pf Recall $NS \bar{X} := A^1(\bar{X})_{alg} = \operatorname{Pic} \bar{X} / \operatorname{Pic}^0 \bar{X}$.

But $\operatorname{Pic}^0 \bar{X}$ is divisible b/c it is an AV.

$\square$

Hence get

(5.1)

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
 & 0 \arrow[d] & \mathrm{Pic}X/m\mathrm{Pic}X \arrow[r] \arrow[d] & (NS(\bar{X})/m\,NS(\bar{X}))^G \arrow[d] & \\
0 \arrow[r] & H^1(\bar{X}, \mu_m)_G \arrow[r] \arrow[d] & H^2(X, \mu_m) \arrow[r] \arrow[d] & H^2(\bar{X}, \mu_m)^G \arrow[r] \arrow[d] & 0 \\
 & \mathrm{Pic}\,\bar{X}[m]_G \arrow[d] & \mathrm{Br}(X)[m] \arrow[r] \arrow[d] & \mathrm{Br}(\bar{X})[m]^G & \\
 & 0 & 0 & &
\end{tikzcd}$$

(Makes contact between $\mathrm{Br}\,X[m]$ and $H^2(X, \mu_m)$ explicit — now we'll use Poincaré duality to get pairing.)

Geometric and Arithmetic Poincaré Duality

Thm Cup product induces perfect pairing of G-modules

$$H^i(\bar{X}, \mu_m) \times H^{4-i}(\bar{X}, \mu_m) \to H^4(X, \mu_m^{\otimes 2}) \xrightarrow{\sim} \mathbb{Z}/m\mathbb{Z}$$

triv. G-mod

Cor (Arithmetic Poincaré duality)

Cup product induces perfect pairing

$$H^i(X, \mu_m) \times H^{5-i}(X, \mu_m) \to H^5(X, \mu_m^{\otimes 2}) \xrightarrow{\sim} \mathbb{Z}/m\mathbb{Z}$$

Rem X is a "5-mfld" b/c k has coh. dim. 1.

Pf Recall SES

$$0 \to H^{i-1}(\bar{X}, A)_G \to H^i(X, A) \to H^i(\bar{X}, A)^G \to 0$$

Take $i=5$, $A = \mu_m^{\otimes 2}$ to get "arithmetic" trace map.

Now given nonzero $\alpha \in H^i(X, \mu_m)$, want to find $\beta \in H^{5-i}(X, \mu_m)$ s.t. $\alpha \cup \beta \neq 0$.

Let $\bar{\alpha} \in H^i(\bar{X}, \mu_m)$ be the image. Then

(i) $\bar{\alpha}$ nonzero. $\exists \bar{\beta} \in H^{4-i}(\bar{X}, \mu_m)$ s.t. $\bar{\alpha} \cup \bar{\beta} \neq 0 \in H^4(\bar{X}, \mu_m)$

Let β be image in $H^{4-i}(\bar{X}, \mu_m)_G$.

<u>Claim</u> β is nonzero

<u>Pf</u> Otherwise $\bar{\beta} = \sigma\gamma - \gamma$ for some γ. But then

$$\bar{\alpha} \cup \bar{\beta} = \bar{\alpha} \cup (\sigma\gamma - \gamma) = \bar{\alpha} \cup \sigma\gamma - \bar{\alpha} \cup \gamma$$

(using $\bar{\alpha} = \sigma\bar{\alpha}$) $\quad = \sigma(\bar{\alpha} \cup \gamma) - \bar{\alpha} \cup \gamma$

(using $\mathbb{Z}/m\mathbb{Z}$ triv G-mod) $\quad = 0$

Now $\beta \in H^{4-i}(\bar{X}, \mu_m)_G$ naturally lives in $H^{5-i}(X, \mu_m)$, done by compatibility w/ $\cup$.

(ii) $\bar{\alpha} = 0$. Then α comes from $H^{i-1}(\bar{X}, \mu_m)_G$;
take $\bar{\beta} \in H^{5-i}(\bar{X}, \mu_m)^G$ which pairs w/ it nontrivially and pick a lift $\square$.

Now we have pairing

$$H^2(X, \mu_m) \times H^3(X, \mu_m) \to H^5(X, \mu_m^{\otimes 2}) \xrightarrow{\sim} \mathbb{Z}/m\mathbb{Z};$$

need to make connection to $Br(X)[m]$.

Bocksteins + the rest

Consider $1 \to \mu_m \to \mu_{m^2} \xrightarrow{\cdot m} \mu_m \to 1$. Induces LES

$$\cdots \to H^{i-1}(X, \mu_m) \xrightarrow{\partial} H^i(X, \mu_m) \to H^i(X, \mu_{m^2}) \to H^i(X, \mu_m) \xrightarrow{\delta} \cdots$$

whence

$$\begin{array}{c} \operatorname{coker}(H^2(X, \mu_{m^2}) \to H^2(X, \mu_m)) \\ \downarrow \wr \; \delta \\ \ker(H^3(X, \mu_m) \to H^3(X, \mu_{m^2})) \end{array}$$

is an isom.

<u>Claim</u> This gp is $Br(X)[m] / m(Br(X)[m^2])$

<u>Pf</u>

$$\begin{tikzcd}
0 \arrow[d] & 0 \arrow[d] & \\
\operatorname{Pic} X / m^2 \operatorname{Pic} X \arrow[r, "\cdot m"] \arrow[d] & \operatorname{Pic} X / m \operatorname{Pic} X \arrow[r] \arrow[d] & 0 \arrow[d] \\
H^2(X, \mu_{m^2}) \arrow[r, "\cdot m"] \arrow[d] & H^2(X, \mu_m) \arrow[r] \arrow[d] & \operatorname{coker} \arrow[d] \\
Br(X)[m^2] \arrow[r, "\cdot m"] \arrow[d] & Br(X)[m] \arrow[r] \arrow[d] & Br(X)[m] / m Br(X)[m^2] \arrow[d] \\
0 & 0 & 0
\end{tikzcd}$$

$\boxed{\xi}$

Let $A := Br(X)[m] / m\, Br(X)[m^2]$

<u>Lemma</u> The pairing $A \times A \to \mathbb{Z}/m\mathbb{Z}$
$(x, y) \mapsto x \cup \delta y$
is perfect + skew-symmetric.

Pf <u>Skew-symmetry</u>: $x \cup \delta y + \delta x \cup x = \delta(x \cup y)$.

But
$$\begin{array}{ccccc} H^4(X, \mu_m^{\otimes 2}) & \xrightarrow{\delta} & H^5(X, \mu_m^{\otimes 2}) & \to & H^5(X, \mu_{m^2}^{\otimes 2}) \\ & & \| & & \| \\ & & \mathbb{Z}/m\mathbb{Z} & \hookrightarrow & \mathbb{Z}/m^2\mathbb{Z} \end{array}$$

is exact, so $\delta(x \cup y) = 0$.

<u>Perfectness</u>:

$$H^2(X, \mu_{m^2}) \to H^2(X, \mu_m) \to A \to 0$$

$$0 \to A \to H^3(X, \mu_m) \to H^3(X, \mu_{m^2})$$

Follows by diagram chase + Poincaré duality $\square$

Passage to the limit gives exactly

$$Br(X)[\ell^\infty] \times Br(X)[\ell^\infty] \to \mathbb{Q}_\ell/\mathbb{Z}_\ell$$

w/ kernel the divisible elts.

(After checking compatibility w/ maps
$Br(X)[m] \to Br(X)[mm']$.)