Local systems of geometric origin

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July 4, 2023

Joint with Josh Lam and Aaron Landesman

Introduction

An open problem about $r \times r$ matrices

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(Katz '96) For all r, iterated version of this: "middle convolution"

Rigid tuples

Upshot (Katz '96)

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Question

What about more general finite orbits of

$$Mod_{0,n} \circlearrowright X_{0,n}(r)?$$

Not all finite orbits are rigid tuples

$$A_1 = \begin{pmatrix} 1 + x_2x_3/x_1 & -x_2^2/x_1 \\ x_3^2/x_1 & 1 - x_2x_3/x_1 \end{pmatrix}, A_2 = \begin{pmatrix} 1 & -x_1 \\ 0 & 1 \end{pmatrix}, A_3 = \begin{pmatrix} 1 & 0 \\ x_1 & 1 \end{pmatrix},$$

$$A_4 = (A_1A_2A_3)^{-1}$$

where

$$x_1 = 2\cos\left(\frac{\pi(\alpha+\beta)}{2}\right),\ x_2 = 2\sin\left(\frac{\pi\alpha}{2}\right),\ x_3 = 2\sin\left(\frac{\pi\beta}{2}\right)$$

for $\alpha, \beta \in \mathbb{Q}$.

Geometric Point of View

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$$\begin{aligned}\mathrm{Mod}_{0,n} &= \pi_0(\mathrm{Homeo}^+(\Sigma_{0,n})) \\ &= \pi_1(\mathscr{M}_{0,n}/S_n) \\ &= \text{“spherical braid group on } n \text{ strands”}\end{aligned}$$

Natural generalization

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Some motivation and conjectures

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A conjecture

Conjecture (Kisin, Whang)

For $g \gg_r 0$, the finite orbits of

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are exactly the representations with finite image.

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are finite orbits of $\mathrm{Mod}_{0,4} \circlearrowright X_{0,4}(2)$ (classified by Lysovyy-Tykhyy (2014), building on work of Schwarz, Poincaré, $\cdots$, Hitchin, Boalch, Doran, Andreev, Kitaev, Dubrovin-Mazzocco, $\cdots$)

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4. Algebraic solutions to:

Schlesinger system, 1912

$$\begin{cases} \dfrac{dA_i}{d\lambda_j} = \dfrac{[A_i, A_j]}{\lambda_i - \lambda_j} & i \neq j \\ \dfrac{dA_i}{d\lambda_i} = -\displaystyle\sum_{j \neq i} \dfrac{[A_i, A_j]}{\lambda_i - \lambda_j} \end{cases}$$

with $A_i \in \mathfrak{sl}_r$, are finite orbits of $\mathrm{Mod}_{0,n} \circlearrowright X_{0,n}(r)$

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(X smooth/f.g. field F with $\mathrm{char}(F) \neq \ell$) An ℓ-adic local system $\mathbb{V}$ on $X_{\overline{F}}$ is of geometric origin if and only if it has finite orbit under the absolute Galois group of F.

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Proposition (Easy direction of non-abelian Tate conjecture)

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These two conjectures contradict each other if $r > 1$!

Some results

Genus 0

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What are the finite orbits of

$$Mod_{0,n} \circlearrowright X_{0,n}(2) = \{(A_1, \cdots, A_n) \in SL_2(k)^n \mid \prod A_i = Id\}?$$

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Theorem (Lam-L–)

In this situation, if some A_i has infinite order, then $(A_1, \cdots, A_n)$ arises via middle convolution from a finite complex reflection group.

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2. For $w \in \mathbb{P}^1 \setminus \{x_1, \cdots, x_n\}$, $\pi^{-1}(w)$ is $\mathbb{Z}/a\mathbb{Z} \times G$ cover of $\mathbb{P}^1$ branched over $w, x_1, \cdots, x_n$, where G is a finite complex reflection group.

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Definition

A group $G \subset GL_r(\mathbb{C})$ is a *finite complex reflection group* if it is finite, acts irreducibly on $\mathbb{C}^r$, and is generated by some g_i such that $\mathrm{rk}(g_i - \mathrm{Id}) = 1$.

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Finite complex reflection groups were classified by Shephard and Todd in 1954! One infinite 3-parameter family and 34 exceptional examples, e.g. classical Weyl groups and automorphism groups of regular polyhedra.

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- Known in rank 2 by Biswas-Gupta-Mj-Whang.

Geometric local systems

Corollary

In the regime (in g, n, r) where these theorems hold, the non-abelian Hodge and Tate conjectures are true for rank r local systems on the generic curve of genus g with n punctures.

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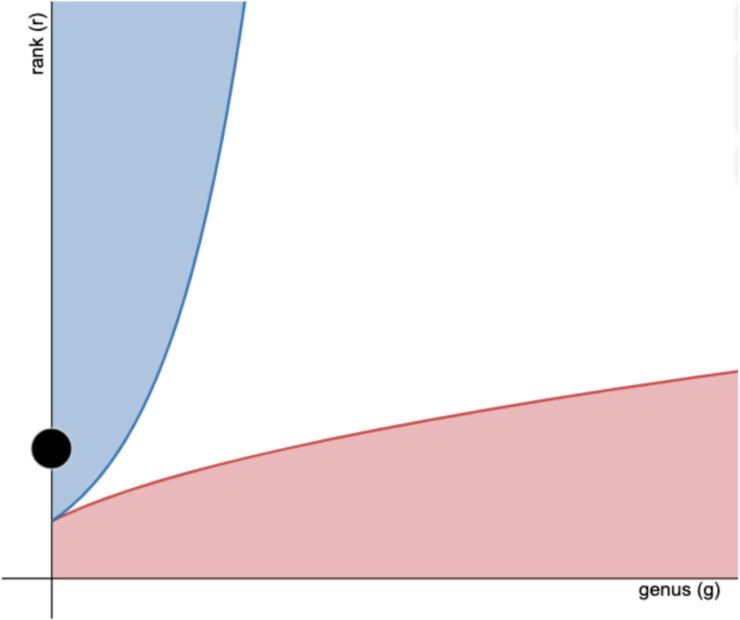
In fact we've written down all geometric local systems (under mild assumptions).

Conjectural picture



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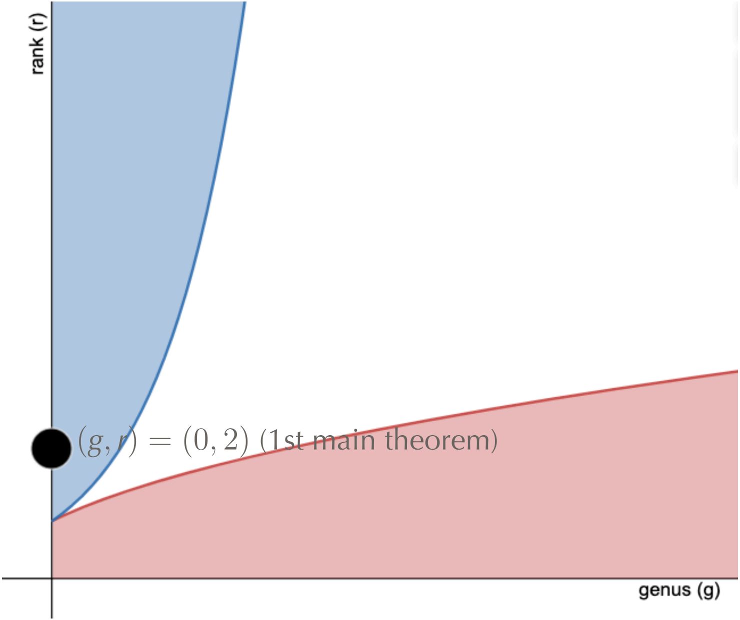

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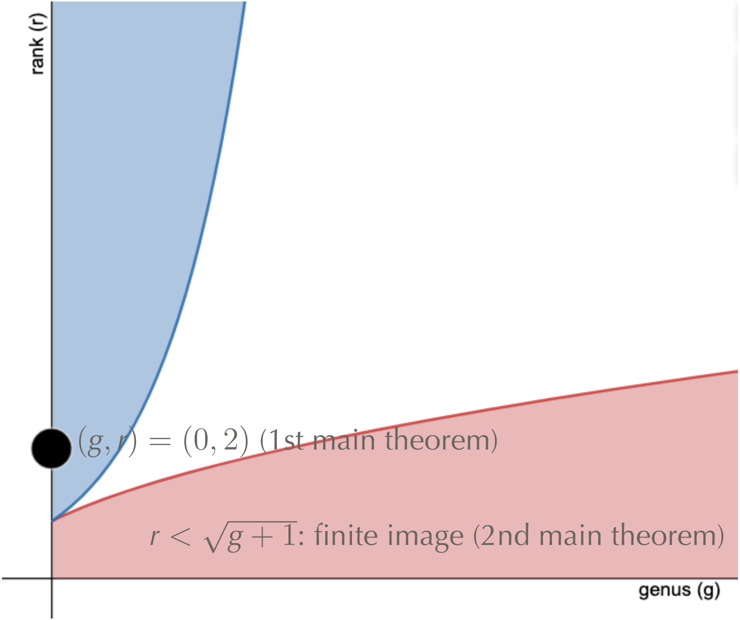

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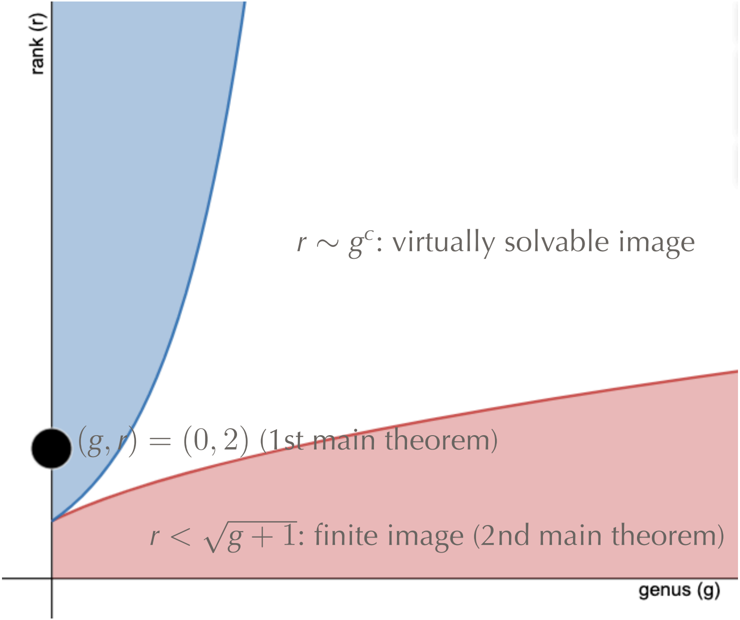

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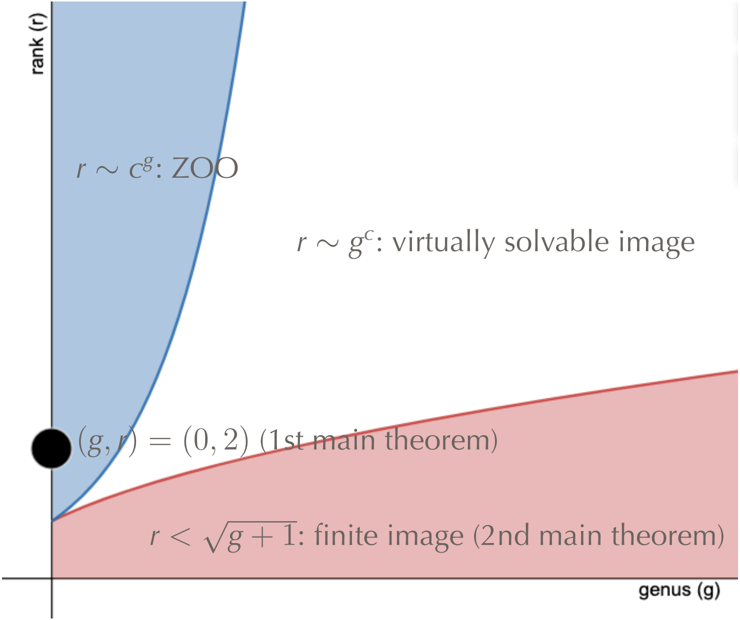

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Conjecture (Superrigidity)

If $g \geq 3$, all irreducible local systems on $\mathscr{M}_{g,n}$ are rigid.

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Assuming Simpson's motivicity conjecture, implies all finite orbits (for $g \geq 3$) are "of geometric origin."

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Question

C a generic curve of genus g with n punctures. Can one write down all local systems on C of geometric origin?

Proof idea

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For simplicity assume ρ irreducible.
$\mathrm{Mod}_{g,n} \cdot [\rho]$ finite $\implies$ there exists:

$$\begin{array}{ccc} \mathscr{C} & & \mathbb{V} \in \mathrm{LocSys}_r(\mathscr{C}) \\ \downarrow & & \\ m \in \mathscr{M} & \xrightarrow{\text{dominant}} & \mathscr{M}_{g,n} \end{array}$$

such that $\mathbb{V}|_{\mathscr{C}_m}$ has monodromy ρ.

Main idea

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- $\mathbb{V}$ defined over $\mathscr{O}_K$ for K a # field.
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The unitary case

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- Rigidity implies $\mathbb{V}|_{\mathscr{C}_m} = \mathbb{V}'|_{\mathscr{C}_m}$, hence ρ is unitary.
- Non-semisimple case: "large g" form of Putman-Wieland conjecture on Prym representations of $\mathrm{Mod}_{g,n} \cdots$

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Superrigidity

For $g \geq 3$, are all local systems on $\Sigma_{g,n}$ with finite orbit under $\mathrm{Mod}_{g,n}$ of geometric origin?

Appendix

Period map computation

$$\begin{array}{ccc} \mathscr{C} & & \mathbb{V} \in \mathrm{LocSys}_R(\mathscr{C}) \\ \downarrow \pi & & \\ m \in \mathscr{M} & \xrightarrow{\text{dominant}} & \mathscr{M}_{g,n} \end{array}$$

Rigidity Theorem (Landesman-L.–)

If $\mathbb{V}|_{\mathscr{C}_m}$ is irreducible and unitary, with $\mathrm{rk}(\mathbb{V}) < \sqrt{g+1}$, then $\mathbb{V}$ is cohomologically rigid.

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- Deformation yields non-trivial kernel, ruled out by Clifford theory.

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C smooth curve of genus g, $\mathbb{V} \in \mathrm{LocSys}_r(C)$ irreducible.

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